

# **Numerical Analysis**

## **MTH603**



**Virtual University of Pakistan**  
**Knowledge beyond the boundaries**

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## Numerical Analysis

### Course Contents

Solution of Non Linear Equations  
 Solution of Linear System of Equations  
 Approximation of Eigen Values  
 Interpolation and Polynomial Approximation  
 Numerical Differentiation  
 Numerical Integration  
 Numerical Solution of Ordinary Differential Equations

### Introduction

We begin this chapter with some of the basic concept of representation of numbers on computers and errors introduced during computation. Problem solving using computers and the steps involved are also discussed in brief.

#### Number (s) System (s)

In our daily life, we use numbers based on the decimal system. In this system, we use ten symbols 0, 1, ..., 9 and the number 10 is called the base of the system.

Thus, when a base  $N$  is given, we need  $N$  different symbols 0, 1, 2, ...,  $(N - 1)$  to represent an arbitrary number.

The number systems commonly used in computers are

| Base, N | Number      |
|---------|-------------|
| 2       | Binary      |
| 8       | Octal       |
| 10      | Decimal     |
| 16      | Hexadecimal |

An arbitrary real number,  $a$  can be written as

$$a = a_m N^m + a_{m-1} N^{m-1} + \cdots + a_1 N^1 + a_0 + a_{-1} N^{-1} + \cdots + a_{-m} N^{-m}$$

In binary system, it has the form,

$$a = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0 + a_{-1} 2^{-1} + \cdots + a_{-m} 2^{-m}$$

The decimal number 1729 is represented and calculated

$$(1729)_{10} = 1 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 9 \times 10^0$$

While the decimal equivalent of binary number 10011001 is

$$\begin{aligned} & 1 \times 2^0 + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4} + 0 \times 2^{-5} + 0 \times 2^{-6} + 1 \times 2^{-7} \\ &= 1 + \frac{1}{8} + \frac{1}{16} + \frac{1}{128} \\ &= (1.1953125)_{10} \end{aligned}$$

Electronic computers use binary system whose base is 2. The two symbols used in this system are 0 and 1, which are called *binary digits* or simply *bits*.

The internal representation of any data within a computer is in binary form. However, we prefer data input and output of numerical results in decimal system. Within the computer, the arithmetic is carried out in binary form.

Conversion of decimal number 47 into its binary equivalent

Sol.

|   |    |                        |
|---|----|------------------------|
| 2 | 47 | Remainder              |
| 2 | 23 | 1                      |
| 2 | 11 | 1                      |
| 2 | 5  | 1                      |
| 2 | 2  | 1                      |
| 2 | 1  | 0                      |
|   | 0  | 1 Most significant bit |

$$(47)_{10} = (101111)_2$$

Binary equivalent of the decimal fraction 0.7625

Sol.

|        |    | Product | Integer |
|--------|----|---------|---------|
| 0.7625 | x2 | 1.5250  | 1       |
| 0.5250 | x2 | 1.0500  | 1       |
| 0.05   | x2 | 0.1     | 0       |
| 0.1    | x2 | 0.2     | 0       |
| 0.2    | x2 | 0.4     | 0       |
| 0.4    | x2 | 0.8     | 0       |
| 0.8    | x2 | 1.6     | 1       |
| 0.6    | x2 | 1.2     | 1       |
| 0.2    | x2 | 0.4     | 0       |

$$(0.7625)_{10} = (0.11 \dots 11(0011))_2$$

Conversion (59)<sub>10</sub> into binary and then into octal.

Sol.

|   |    |   |
|---|----|---|
| 2 | 29 | 1 |
| 2 | 14 | 1 |
| 2 | 7  | 0 |
| 2 | 3  | 1 |
| 2 | 1  | 1 |
|   | 0  | 1 |

---

|

$$(59)_{10} = (11011)_2$$

$$(111011)_2 = 111011 = (73)_8$$

## Errors in Computations

Numerically, computed solutions are subject to certain errors. It may be fruitful to identify the error sources and their growth while classifying the errors in numerical computation. These are

***Inherent errors,***

***Local round-off errors***

***Local truncation errors***

**Inherent errors**

It is that quantity of error which is present in the statement of the problem itself, before finding its solution. It arises due to the simplified assumptions made in the mathematical modeling of a problem. It can also arise when the data is obtained from certain physical measurements of the parameters of the problem.

***Local round-off errors***

Every computer has a finite word length and therefore it is possible to store only a fixed number of digits of a given input number. Since computers store information in binary form, storing an exact decimal number in its binary form into the computer memory gives an error. This error is computer dependent.

At the end of computation of a particular problem, the final results in the computer, which is obviously in binary form, should be converted into decimal form-a form understandable to the user-before their print out. Therefore, an additional error is committed at this stage too.

This error is called *local round-off* error.

$$(0.7625)_{10} = (0.110000110011)_2$$

If a particular computer system has a word length of 12 bits only, then the decimal number 0.7625 is stored in the computer memory in binary form as 0.110000110011. However, it is equivalent to 0.76245.

Thus, in storing the number 0.7625, we have committed an error equal to 0.00005, which is the round-off error; inherent with the computer system considered.

Thus, we define the *error* as

$$\text{Error} = \text{True value} - \text{Computed value}$$

*Absolute* error, denoted by  $|\text{Error}|$ ,

While, the *relative error* is defined as

$$\text{Relative error} = \frac{|\text{Error}|}{|\text{True value}|}$$

***Local truncation error***

It is generally easier to expand a function into a power series using Taylor series expansion and evaluate it by retaining the first few terms. For example, we may approximate the function  $f(x) = \cos x$  by the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

If we use only the first three terms to compute  $\cos x$  for a given  $x$ , we get an approximate answer. Here, the error is due to truncating the series. Suppose, we retain the first  $n$  terms, the truncation error (TE) is given by

$$TE \leq \frac{x^{2n+2}}{(2n+2)!}$$

The TE is independent of the computer used.

If we wish to compute  $\cos x$  for accurate with five significant digits, the question is, how many terms in the expansion are to be included? In this situation

$$\frac{x^{2n+2}}{(2n+2)!} < .5 \times 10^{-5} = 5 \times 10^{-6}$$

Taking logarithm on both sides, we get

$$(2n+2) \log x - \log[(2n+2)!]$$

$$< \log_{10} 5 - 6 \log_{10} 10 = 0.699 - 6 = -5.3$$

or

$$\log[(2n+2)!] - (2n+2) \log x > 5.3$$

We can observe that, the above inequality is satisfied for  $n = 7$ . Hence, seven terms in the expansion are required to get the value of  $\cos x$ , with the prescribed accuracy

The truncation error is given by

$$TE \leq \frac{x^{16}}{16!}$$

**Polynomial**

An expression of the form  $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$  where  $n$  is a positive integer and  $a_0, a_1, a_2, \dots, a_n$  are real constants, such type of expression is called an  $n$ th degree polynomial in  $x$  if  $a_0 \neq 0$

**Algebraic equation:**

An equation  $f(x)=0$  is said to be the algebraic equation in  $x$  if it is purely a polynomial in  $x$ .

For example

$x^5 + x^4 + 3x^2 + x - 6 = 0$  It is a fifth order polynomial and so this equation is an algebraic equation.

$$x^3 - 6 = 0$$

$$x^6 - 7x = 0$$

$$y^4 - 4y^3 + 3y^2 - y - 2 = 0 \text{ polynomial in } y$$

$$t^4 - 6t^2 - 21 = 0 \text{ polynomial in } t$$

These all are the examples of the polynomial or algebraic equations.

**Some facts**

1. Every equation of the form  $f(x)=0$  has at least one root, it may be real or complex.
2. Every polynomial of  $n$ th degree has  $n$  and only  $n$  roots.
3. If  $f(x)=0$  is an equation of odd degree, then it has at least one real root whose sign is opposite to that of last term.
4. If  $f(x)=0$  is an equation of even degree whose last term is negative then it has at least one positive and at least one negative root.

**Transcendental equation**

An equation is said to be transcendental equation if it has logarithmic, trigonometric and exponential function or combination of all these three.

For example

$$e^x - 5x - 3 = 0 \text{ it is a transcendental equation as it has an exponential function}$$

$$e^x - \sin x = 0$$

$$\ln x - \sin x = 0$$

$$2 \sec^2 x - \tan x - e^x = 0$$

These all are the examples of transcendental equation.

**Root of an equation**

For an equation  $f(x)=0$  to find the solution we find such value which satisfy the equation  $f(x)=0$ , these values are known as the roots of the equation.

A value  $a$  is known as the root of an equation  $f(x)=0$  if and only if  $f(a)=0$ .

### Properties of an Algebraic equation

1. Complex roots occur in the pairs. That is, If  $(a+ib)$  is a root of  $f(x)=0$  then  $(a-ib)$  is also a root of the equation
2. if  $x=a$  is a root of the equation  $f(x)=0$  a polynomial of  $n$ th degree, then  $(x-a)$  is a factor of  $f(x)$  and by dividing  $f(x)$  by  $(x-a)$  we get a polynomial of degree  $n-1$ .

### Descartes rule of signs

This rule shows the relation ship between the signs of coefficients of an equation and its roots.

“The number of positive roots of an algebraic equation  $f(x)=0$  with real coefficients can not exceed the number of changes in the signs of the coefficients in the polynomial  $f(x)=0$ . similarly the number of negative roots of the equation can not exceed the number of changes in the sign of coefficients of  $f(-x)=0$ ”

Consider the equation  $x^3 - 3x^2 + 4x - 5 = 0$  here it is an equation of degree three and there are three changes in the signs

First +ve to -ve second -ve to +ve and third +ve to -ve so the tree roots will be positive

Now  $f(-x) = -x^3 - 3x^2 - 4x - 5$  so there is no change of sign so there will be no negative root of this equation.

### Intermediate value property

If  $f(x)$  is a real valued continuous function in the closed interval  $a \leq x \leq b$  if  $f(a)$  and  $f(b)$  have opposite signs once; that is  $f(x)=0$  has at least one root  $\beta$  such that  $a \leq \beta \leq b$

Simply

If  $f(x)=0$  is a polynomial equation and if  $f(a)$  and  $f(b)$  are of different signs, then  $f(x)=0$  must have at least one real root between  $a$  and  $b$ .

Numerical methods for solving either algebraic or transcendental equation are classified into two groups

### Direct methods

Those methods which do not require any information about the initial approximation of root to start the solution are known as direct methods.

The examples of direct methods are Graefee root squaring method, Gauss elimination method and Gauss Jordan method. All these methods do not require any type of initial approximation.

### Iterative methods

These methods require an initial approximation to start.

Bisection method, Newton raphson method, secant method, jacobie method are all examples of iterative methods.

### How to get an initial approximation?

The initial approximation may be found by two methods either by graphical method or analytical method

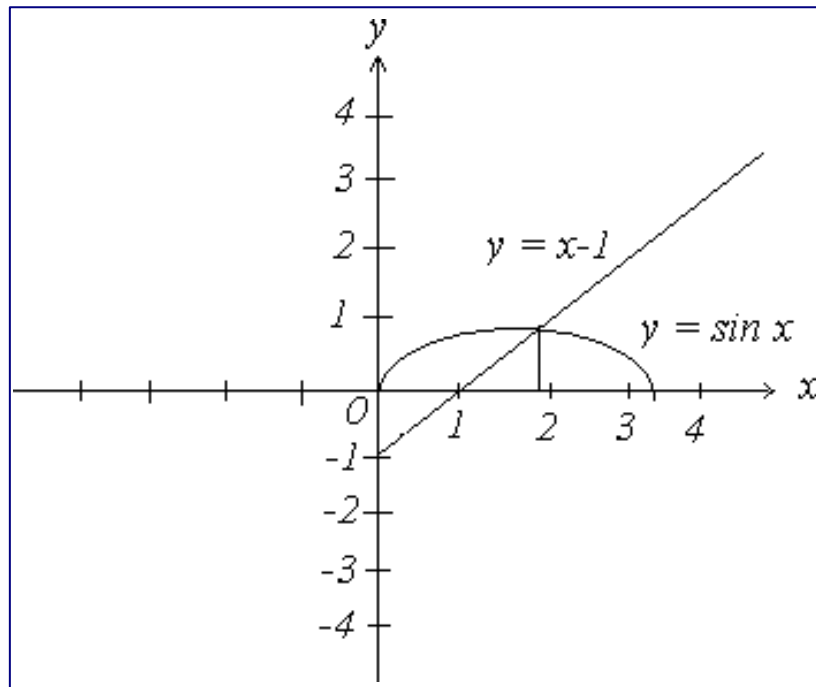
Graphical method

The equation  $f(x)=0$  can be rewritten as  $f_1(x) = f_2(x)$  and initial approximation of  $f(x)$  may be taken as the abscissa of the point of intersection of graphs of

$y = f_1(x)$  and  $y = f_2(x)$

for example  $f(x) = x - \sin x - 1 = 0$

so this may be written as  $x - 1 = \sin x$  Now we shall draw the graphs of  $y = x - 1$  and  $y = \sin x$



Here both the graphs cut each other at 1.9 so the initial approximation should be taken as 1.9

### Analytical method

This method is based on the intermediate value property in this we locate two values  $a$  and  $b$  such that  $f(a)$  and  $f(b)$  have opposite signs then we use the fact that the root lies

between both these points ,this method can be used for both transcendental and algebraic equations.

Consider the equation

$$f(x) = 3 - \sqrt{1 + \sin x} = 0$$

$$f(1) = 3 - \sqrt{1 + \sin(1 \times \frac{180}{\pi})} = 3 - \sqrt{1 + 0.84147} = 1.64299$$

Here  $f(0)$  and  $f(1)$  are of opposite signs making use of intermediate value property we infer that one root lies between 0 and 1 .

So in analytical method we must always start with an initial interval (a,b) so that  $f(a)$  and  $f(b)$  have opposite signs.

### Bisection method (Bolzano)

Suppose you have to locate the root of the equation  $f(x)=0$  in an interval say  $(x_0, x_1)$ , let  $f(x_0)$  and  $f(x_1)$  are of opposite signs such that  $f(x_0)f(x_1) < 0$

Then the graph of the function crossed the x-axis between  $x_0$  and  $x_1$  which exists the existence of at least one root in the interval  $(x_0, x_1)$ .

The desired root is approximately defined by the mid point  $x_2 = \frac{x_0 + x_1}{2}$  if  $f(x_2) = 0$  then

$x_2$  is the root of the equation otherwise the root lies either between  $x_0$  and  $x_2$  or  $x_1$  and  $x_2$

Now we define the next approximation by  $x_3 = \frac{x_0 + x_2}{2}$  provided  $f(x_0)f(x_2) < 0$  then

root may be found between  $x_0$  and  $x_2$

If provided  $f(x_1)f(x_2) < 0$  then root may be found between  $x_1$  and  $x_2$  by  $x_3 = \frac{x_1 + x_2}{2}$

Thus at each step we find the desired root to the required accuracy or narrow the range to half the previous interval.

This process of halving the intervals is continued in order to get smaller and smaller interval within which the desired root lies. Continuation of this process eventually gives us the required root.

**NOTE: In numerical analysis all the calculation are carried out in radians mode and the value of pi is supposed to be 3.14**

### Example

Solve  $x^3 - 9x + 1 = 0$  for the root between  $x=2$  and  $x=4$  by bisection method

### Solution:

Here we are given the interval (2,4) so we need not to carry out intermediate value property to locate initial approximation.

Here

$$f(x) = x^3 - 9x + 1 = 0$$

$$\text{now } f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = 9$$

$$f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 29$$

here  $f(2) f(3) < 0$  so root lies between 2 and 4

$$x_0 = 2 \quad x_1 = 4$$

$$x_2 = \frac{2+4}{2} = 3$$

$$f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1$$

here  $f(2) f(3) < 0$  so the root lies between 2 and 3

$$x_3 = \frac{2+3}{2} = 2.5$$

$$f(2.5) = 2.5^3 - 9(2.5) + 1 = 15.625 - 22.5 + 1 = -5.875 < 0$$

so the root lies between 2.5 and 3 as  $f(2.5) f(3) < 0$

$$x_4 = \frac{2.5+3}{2} = 2.75$$

now

similarly  $x_5 = 2.875$  and  $x_6 = 2.9375$  and the process is continued until the desired accuracy is obtained.

| $n$ | $x_n$  | $f(x_n)$ |
|-----|--------|----------|
| 2   | 3      | 1.0      |
| 3   | 2.5    | -5.875   |
| 4   | 2.75   | -2.9531  |
| 5   | 2.875  | -1.1113  |
| 6   | 2.9375 | -0.0901  |

### When to stop the process of iteration?

Here in the solved example neither accuracy is mentioned and nor number of iteration is mentioned by when you are provided with the number of iteration then you will carry those number of iteration and will stop but second case is accuracy like you are asked to

find root with an accuracy of  $10^{-3}$  then you will check the accuracy by subtracting two consecutive root like 2.135648 and 2.135769

$$2.135769 - 2.135648 = 0.000121$$

So the accuracy of  $10^{-3}$  is achieved so you will stop here.

### Example:

Carry out the five iterations for the function  $f(x) = 2x \cos(2x) - (x+1)^2$

**Note: All the calculations should be done in the radians.**

**Solution:**

$$f(x) = 2x \cos(2x) - (x+1)^2$$

$$f(-1) = 2(-1) \cos(-2) - (-1+1)^2 = -2(-0.4161) = +0.8322 > 0$$

$$f(0) = 2(0) \cos(0) - (0+1)^2 = -1 = -1 < 0$$

so the root lies between 0 and -1 as  $f(0)f(-1) < 0$

$$x_2 = \frac{0-1}{2} = -0.5$$

$$f(-0.5) = 2(-0.5) \cos(-1) - (-0.5+1)^2 = -0.5403 - 0.25 = -0.7903 < 0$$

so root lies between -1 and -0.5 as  $f(-1)f(-0.5)$

$$x_3 = \frac{-0.5-1}{2} = -0.75$$

$$f(-0.75) = 2(-0.75) \cos(-1.5) - (-0.75+1)^2 = -0.106 - 0.0625 = -0.1686 < 0$$

so root lies between -1 and -0.75 as  $f(-1)f(-0.75)$

$$x_4 = \frac{-0.75-1}{2} = -0.875$$

$$f(-0.875) = 2(-0.875) \cos(-1.75) - (-0.875+1)^2 = 0.3119 - 0.015625 = 0.296275 > 0$$

so root lies between -0.875 and -0.75 as  $f(-0.75)f(-0.875)$

$$x_5 = \frac{-0.75-0.875}{2} = -0.8125$$

$$f(-0.8125) = 2(-0.8125) \cos(-1.625) - (-0.8125+1)^2 = 0.0880 - 0.0351 = 0.052970 > 0$$

so root lies between -0.8125 and -0.75 as  $f(-0.75)f(-0.8125)$

$$x_5 = \frac{-0.75-0.8125}{2} = -0.78125$$

### Example :

Carry out the first five iterations  $f(x) = x \cos x - 2x^2 + 3x - 1$  ,  $1.2 \leq x \leq 1.3$

**Note: All the calculations should be done in the radians.**

**Solution:**

$$f(x) = x \cos x - 2x^2 + 3x - 1$$

$$\begin{aligned} f(1.2) &= 1.2 \cos 1.2 - 2(1.2)^2 + 3(1.2) - 1 \\ &= 1.2(0.3623) - 2(1.44) + 3.6 - 1 = 0.1548 > 0 \end{aligned}$$

$$\begin{aligned} f(1.3) &= 1.3 \cos 1.3 - 2(1.3)^2 + 3(1.3) - 1 \\ &= 1.3(0.2674) - 2(1.69) + 3.9 - 1 = -0.1322 < 0 \end{aligned}$$

as  $f(1.2)f(1.3) < 0$  so the root lies between both

$$x_2 = \frac{1.2+1.3}{2} = 1.25$$

$$\begin{aligned} f(1.25) &= 1.25 \cos 1.25 - 2(1.25)^2 + 3(1.25) - 1 \\ &= 1.25(0.3153) - 2(1.5625) + 3.75 - 1 = 0.0191 > 0 \end{aligned}$$

as  $f(1.25)f(1.3) < 0$  so the root lies between both

$$x_3 = \frac{1.25+1.3}{2} = 1.275$$

$$\begin{aligned} f(1.275) &= 1.275 \cos 1.275 - 2(1.275)^2 + 3(1.275) - 1 \\ &= 1.275(0.2915) - 2(1.6256) + 3.825 - 1 = -0.0545 < 0 \end{aligned}$$

as  $f(1.25)f(1.275) < 0$  so the root lies between both

$$x_4 = \frac{1.25+1.275}{2} = 1.2625$$

$$\begin{aligned} f(1.2625) &= 1.2625 \cos 1.2625 - 2(1.2625)^2 + 3(1.2625) - 1 \\ &= 1.2625(0.3034) - 2(1.5939) + 3.7875 - 1 = -0.0172 < 0 \end{aligned}$$

as  $f(1.25)f(1.2625) < 0$  so the root lies between both

$$x_5 = \frac{1.25+1.2625}{2} = 1.25625$$

$$\begin{aligned} f(1.25625) &= 1.25625 \cos 1.25625 - 2(1.25625)^2 + 3(1.25625) - 1 \\ &= 1.25625(0.3093) - 2(1.5781) + 3(1.25625) - 1 = 0.00108 > 0 \end{aligned}$$

as  $f(1.25625)f(1.2625) < 0$  so the root lies between both

$$x_6 = \frac{1.25625+1.2625}{2} = 1.259375$$

### Regula-Falsi method (Method of false position)

Here we choose two points  $x_n$  and  $x_{n-1}$  such that  $f(x_n)$  and  $f(x_{n-1})$  have opposite signs. Intermediate value property suggests that the graph of the  $y=f(x)$  crosses the x-axis between these two points and therefore, a root lies between these two points. Thus to find the real root of  $f(x)=0$  using Regula-Falsi method, we replace the part of the curve between the points  $A(x_n, f(x_n))$  and  $B(x_{n-1}, f(x_{n-1}))$  by a chord in the interval and we take the point of intersection of this chord with x-axis as initial approximation. Now, the equation of the chord joining the points A and B is,

$$\frac{y - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n}$$

Setting  $y=0$  in the above equation we get

$$x = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Hence the first approximation to the root is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

We observe that  $f(x_{n-1})$  and  $f(x_{n+1})$  are of opposite signs thus it is possible to apply the above procedure, to determine the line through B and  $A_1$  and so on.

Hence for successive approximation to the root above formula is used.

#### Example

Use the Regula-Falsi method to compute a real root of the equation  $x^3 - 9x + 1 = 0$ ,

- (i) if the root lies between 2 and 4
- (ii) if the root lies between 2 and 3.

Comment on the results.

#### Solution

Let

$$f(x) = x^3 - 9x + 1$$

$$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9 \text{ and } f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 29.$$

Since  $f(2)$  and  $f(4)$  are of opposite signs, the root of  $f(x) = 0$  lies between 2 and 4.

Taking  $x_1 = 2$ ,  $x_2 = 4$  and using Regula-Falsi method, the first approximation is given by

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 4 - \frac{4 - 2}{29 - (-9)} (29) = 4 - \frac{2(29)}{38} \\ &= 4 - \frac{58}{38} = 4 - 1.5263 = 2.4736 \end{aligned}$$

Now

$$f(x_3) = 2.4736^3 - 9(2.4736) + 1 = 15.13520 - 22.2624 + 1 = -6.12644.$$

Since  $f(x_2)$  and  $f(x_3)$  are of opposite signs, the root lies between  $x_2$  and  $x_3$ .

The second approximation to the root is given as

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.4736 - \frac{2.4736 - 2}{-6.12644 - 29} (-6.12644) \\ &= 2.4736 - \frac{-1.5264}{-35.12644} (-6.12644) = 2.4736 - (0.04345)(-6.12644) \\ &= 2.4736 + 0.26619 = 2.73989 \end{aligned}$$

Therefore

$$f(x_4) = 2.73989^3 - 9(2.73989) + 1 = 20.5683 - 24.65901 + 1 = -3.090707.$$

Now, since  $f(x_2)$  and  $f(x_4)$  are of opposite signs, the third approximation is obtained from

$$\begin{aligned} x_5 &= x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.73989 - \frac{2.73989 - 2}{-3.090707 - 29} (-3.090707) = 2.86125 \\ &= 2.73989 - \frac{-1.26011}{-32.090707} (-3.090707) = 2.73989 + 0.039267(3.090707) = 2.73989 + 0.121363 = 2.86125 \end{aligned}$$

Now

$$f(x_5) = 2.86125^3 - 9(2.86125) + 1 = 23.42434 - 25.75125 + 1 = -1.326868.$$

(ii)

Here

$$f(x) = x^3 - 9x + 1$$

$$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9 \text{ and } f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1.$$

Since  $f(2)$  and  $f(3)$  are of opposite signs, the root of  $f(x) = 0$  lies between 2 and 3.

Taking  $x_1 = 2$ ,  $x_2 = 3$  and using Regula-Falsi method, the first approximation is given by

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{3 - 2}{1 - (-9)} (-9) \\ &= 3 - \frac{1}{10} = 2.9 \end{aligned}$$

$$f(x_3) = 2.9^3 - 9(2.9) + 1 = 24.389 - 26.1 + 1 = -0.711$$

Since  $f(x_2)$  and  $f(x_3)$  are of opposite signs, the root lies between  $x_2$  and  $x_3$ .

The second approximation to the root is given as

$$\begin{aligned} x_4 &= 2.9 - \frac{2.9 - 2}{-0.711 - (-9)} (-0.711) = 2.9 - \frac{-0.1}{-1.711} (-0.711) = 2.9 - \frac{-0.1}{-1.711} (-0.711) \\ &= 2.9 - (0.05844)(-0.711) = 2.9 + 0.04156 = 2.94156 \end{aligned}$$

$$f(x_4) = -0.0207$$

$$f(x_4) = 2.94156^3 - 9(2.94156) + 1 = 25.45265 - 26.47404 + 1 = -0.0207$$

Now, we observe that  $f(x_2)$  and  $f(x_4)$  are of opposite signs; the third approximation is obtained from

$$x_5 = 2.94156 - \frac{2.94156 - 3}{-0.0207 - 1}(-0.0207) = 2.94156 - \frac{-0.05844}{-1.0207}(-0.0207) \\ = 2.94156 - (-0.05725)(-0.0207) = 2.94275$$

$$f(x_5) = 2.94275^3 - 9(2.94275) + 1 = 25.48356 - 26.48475 + 1 = -0.0011896$$

We observe that the value of the root as a third approximation is evidently different in both the cases, while the value of  $x_5$ , when the interval considered is  $(2, 3)$ , is closer to the root.

**Important observation: The initial interval  $(x_1, x_2)$  in which the root of the equation lies should be sufficiently small.**

Example

Use Regula-Falsi method to find a real root of the equation  $\ln x - \cos x = 0$  accurate to four decimal places after three successive approximations.

**Note: here is a transcendental equation all the calculation should be done in the radians mode**

Sol:

$$f(x) = \ln x - \cos x$$

we have

$$f(1) = \ln 1 - \cos(1) = 0 - 0.540302 = -0.540302 < 0$$

$$f(2) = \ln 2 - \cos(2) = 0.69315 - 0.41615 = 1.109$$

As  $f(1)f(2) < 0$  so the root lies between 1 and 2

the first approximation is *obtained form*

$$x_3 = 2 - \frac{2 - 1}{1.109 + 0.540302}(1.109) \\ = 2 - \frac{1.1093}{1.6496} = 1.3275$$

$$f(x_3) = \ln 1.3275 - \cos 1.3275 = 0.2833 - 0.2409 = 0.0424$$

Now, since  $f(x_1)$  and  $f(x_3)$  are of opposite signs, the second approximation is obtained as

$$x_4 = 1.3275 - \frac{(1.3275)(0.0424)}{0.0424 + 0.5403} \\ = 1.3037$$

$$f(x_4) = \ln 1.3037 - \cos 1.3037 = 1.24816 \times 10^{-3}$$

Similarly, we observe that  $f(x_1)$  and  $f(x_4)$  are of opposite signs, so, the third approximation is given by

$$\begin{aligned}x_5 &= 1.3037 - \frac{(0.3037)(0.001248)}{0.001248 + 0.5403} \\&= 1.3030 \\f(x_5) &= 0.6245 \times 10^{-4}\end{aligned}$$

The required real root is 1.3030

### Example:

Use method of false position to solve  $e^{-x} + 2^{-x} + 2 \cos x - 6 = 0 \quad 1 \leq x \leq 2$

### Solution:

$$f(x) = e^{-x} + 2^{-x} + 2 \cos x - 6$$

$$x_0 = 1, \quad x_1 = 2$$

now

$$x_{n+1} = \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$f(1) = e^{-1} + 2^{-1} + 2 \cos 1 - 6 = 2.7182 + 0.5 + 2(0.5403) - 6 = -1.7011$$

$$f(2) = e^{-2} + 2^{-2} + 2 \cos 2 - 6 = 7.3886 + 0.25 + 2(-0.4161) - 6 = 0.8068$$

now for  $n = 1$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 2 - \frac{2-1}{0.8068+1.7011} (0.8068)$$

$$x_2 = 2 - \frac{1}{2.5079} (0.8068) = 1.6783$$

$$f(1.6783) = e^{-1.6783} + 2^{-1.6783} + 2 \cos(1.6783) - 6 = -0.5457$$

now for  $n = 2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 1.6783 - \frac{1.6783 - 2}{(-0.5457) - 0.8068} (-0.5457)$$

$$x_3 = 1.6783 - \frac{(-0.3217)}{(-1.3525)} (-0.5457) = 1.6783 + 0.12979 = 1.8081$$

$$f(1.8081) = e^{1.6783} + 2^{-1.8081} + 2 \cos(1.8081) - 6 = -0.8575$$

now for  $n = 3$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 1.8081 - \frac{1.8081 - 1.6783}{(-0.08575) + 0.5457} (-0.08575)$$

$$x_3 = 1.8081 - \frac{0.1298}{0.45995} (-0.08575) = 1.6783 + 0.12979 = 1.8323$$

$$f(1.8323) = e^{1.8323} + 2^{-1.8323} + 2 \cos(1.8323) - 6 = 0.1199$$

now for  $n = 4$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 1.8323 - \frac{1.8323 - 1.8081}{0.01199 + 0.08575} (0.01199)$$

$$x_5 = 1.8323 - \frac{0.0242}{0.09774} (0.01199) = 1.8323 - 0.00296 = 1.8293$$

$$f(1.8293) = e^{1.8293} + 2^{-1.8293} + 2 \cos(1.8293) - 6 = -0.000343$$

now for  $n = 5$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5) = 1.8293 - \frac{1.8293 - 1.8323}{-0.000343 - 0.01199} (-0.000343)$$

$$x_6 = 1.8293 - \frac{(-0.003)}{-0.012333} (-0.000343) = 1.8293$$

### Example:

Solve the equation  $2x \cos 2x - (x - 2)^2 = 0$   $2 \leq x \leq 3$  Perform only three iterations.

### Solution

$$f(x) = 2x \cos 2x - (x - 2)^2$$

here  $x_0 = 2$  and  $x_1 = 3$

$$\text{so } f(2) = 2(2) \cos 4 - (2 - 2)^2 = 4 \cos 4 = -2.6146$$

$$f(3) = 2(3) \cos 2(3) - (3 - 2)^2 = 6 \cos 6 - 1 = 4.7610$$

$$\text{here } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

for  $n = 1$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{3 - 2}{4.7610 - (-2.6146)} (4.7610)$$

$$= 3 - \frac{1}{7.3756}(4.7610) = 3 - 0.6455 = 2.3545$$

$$f(2.3545) = 2(2.3545) \cos 2(2.3545) - (2.3545 - 2)^2 = 4.709 \cos 4.709 - 0.1257 = -0.1416$$

for  $n = 2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.3545 - \frac{2.3545 - 3}{-0.1416 - 4.7610}(-0.1416) \\ = 2.3731$$

$$f(2.3713) = 2(2.3713) \cos 2(2.3713) - (2.3713 - 2)^2 = 4.7462 \cos 4.7462 - 0.1392 = -0.1392$$

for  $n = 3$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.3713 - \frac{2.3731 - 2.3545}{0.0212 + 0.1416}(0.0212) \\ = 2.3707$$

$$f(2.3707) = 2(2.3707) \cos 2(2.3707) - (2.3707 - 2)^2 = 4.7414 \cos 4.7412 - 0.1374 = 0.00013$$

for  $n = 4$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 2.3707 - \frac{2.3707 - 2.3731}{0.00013 - 0.0212}(0.00013) \\ = 2.3707$$

**Example**

Using Regula-Falsi method, find the real root of the following equation correct, to three decimal places:  $x \log_{10} x = 1.2$

Solution:

$$\text{Let } f(x) = x \log_{10} x - 1.2$$

$$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979,$$

$$f(3) = 3 \log_{10} 3 - 1.2 = 0.2314.$$

Since  $f(2)$  and  $f(3)$  are of opposite signs, the real root lies between  $x_1 = 2$ ,  $x_2 = 3$ .

The first approximation is obtained from

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{3 - 2}{0.2314 + 0.5979} (0.2314) \\ &= 3 - \frac{0.2314}{0.8293} = 2.72097 \end{aligned}$$

$$f(x_3) = \text{Let } f(x) = 2.72097 \log_{10} 2.72097 - 1.2 = -0.01713.$$

Since  $f(x_2)$  and  $f(x_3)$  are of opposite signs, the root of  $f(x) = 0$  lies between  $x_2$  and  $x_3$ . Now, the second approximation is given by

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.72097 - \frac{2.72097 - 3}{-0.01713 - 0.2314} (-0.01713) = 2.7402 \\ f(x_4) &= 2.7402 \log_{10} 2.7402 - 1.2 = -3.8905 \times 10^{-4} \end{aligned}$$

Thus, the root of the given equation correct to three decimal places is 2.740

**NOTE:** Here if TOL is given then we can simply find the value of TOL by subtracting both the consecutive roots and write it in the exponential notation if the required TOL is obtained then we stop.

**Method of Iteration**

Method of iterations can be applied to find a real root of the equation  $f(x) = 0$  by rewriting the same in the form.

$$x = \phi(x)$$

Let  $x = x_0$  be the initial approximation to the actual root, say,  $\alpha$  of the equation. Then the first approximation is  $x_1 = \phi(x_0)$  and the successive approximations are  $x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$ ,  $x_4 = \phi(x_3)$ , ...,  $x_n = \phi(x_{n-1})$  if sequence of the approximate roots,  $x_1, x_2, x_3, \dots, x_n$  converges to  $\alpha$  it is taken as the root of the equation  $f(x) = 0$ .

For convergence purpose the initial approximation is to be done carefully. The choice of the  $x_0$  is done according to the theorem.

**Theorem**

If  $\alpha$  be a root of  $f(x)=0$  which is equivalent to  $x=\phi(x)$   $I$  be any interval containing the point  $x=\alpha$  and  $|\phi'(x)|<1 \forall x \in I$ , then the sequence of approximations  $x_1, x_2, x_3, \dots, x_n$  will converge to the root  $\alpha$  provided that the initial approximation  $x_0$  is chosen in  $I$

Example,

$$f(x) = \cos x - 2x + 3 = 0.$$

It can be

$$\text{Rewritten as } x = \frac{1}{2}(\cos x + 3) = \phi(x)$$

$$\phi(x) = \frac{1}{2}(\cos x + 3)$$

$$f(x) = \cos x - 2x + 3 = 0.$$

$$f(1) = \cos 1 - 2(1) + 3 = 1.54030 > 0$$

$$f(2) = \cos 2 - 2(2) + 3 = -0.041614 - 4 + 3 = -1.41614 < 0$$

so root lies between 1 and 2

$$\phi'(x) = -\frac{1}{2}(\sin x)$$

both  $\phi'(1)$  and  $\phi'(2) < 1$  so the method of iterations can be applied

let  $x_0 = 1.5$

$$x_1 = \frac{1}{2}(\cos x_0 + 3) = \frac{1}{2}(\cos(1.5) + 3) = 1.999825$$

$$x_2 = \frac{1}{2}(\cos x_1 + 3) = \frac{1}{2}(\cos(1.999825) + 3) = 1.999695$$

$$x_3 = \frac{1}{2}(\cos x_2 + 3) = \frac{1}{2}(\cos(1.999625) + 3) = 1.999695$$

So this is the required root correct up to 5 places of decimal.

**Example**

Find the real root of the equation  $x^3 + x^2 - 1 = 0$  by method of iterations

Solution

$$\text{let } f(x) = x^3 + x^2 - 1$$

$$\text{now } f(0) = 0^3 + 0^2 - 1 = -1 < 0$$

$$f(1) = 1^3 + 1^2 - 1 = 1 > 0$$

hence a real root lies between 0 and 1

here

$$x^3 + x^2 - 1 = 0$$

$$x^2(x+1) = 1$$

$$x^2 = \frac{1}{(x+1)} \Rightarrow x = \frac{1}{\sqrt{x+1}} = \phi(x)$$

$$\text{here } \phi'(x) = -\frac{1}{2} \left[ \frac{1}{(x+1)^2} \right]$$

$$\phi'(0) = 1/2 < 1 \text{ and } \phi'(1) = 1/2^2 < 1$$

so  $\phi'(x) < 1$  for all the values in the interval

let  $x_0 = 0.65$

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{1.65}} = 0.7784989$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{1.7784989}} = 0.7498479$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{1.7498479}} = 0.7559617$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{1.7559617}} = 0.7546446$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{1.7546446}} = 0.7549278$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{1.7549278}} = 0.7548668$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{x_6+1}} = \frac{1}{\sqrt{1.7548668}} = 0.7548799$$

$$x_8 = \phi(x_7) = \frac{1}{\sqrt{x_7+1}} = \frac{1}{\sqrt{1.7548799}} = 0.7548771$$

$$x_9 = \phi(x_8) = \frac{1}{\sqrt{x_8+1}} = \frac{1}{\sqrt{1.7548771}} = 0.7548777$$

$$x_{10} = \phi(x_9) = \frac{1}{\sqrt{x_9+1}} = \frac{1}{\sqrt{1.7548777}} = 0.7548776$$

$$x_{11} = \phi(x_{10}) = \frac{1}{\sqrt{x_{10}+1}} = \frac{1}{\sqrt{1.7548776}} = 0.7548776$$

hence root is 0.7548776

Note: In this question the accuracy up to 7 places is acquired or here the TOL is  $10^{-7}$

**Example**

Find a real root of the equation  $\cos x = 3x - 1$  correct to seven places of decimal.

**Solution**

Here it is a transcendental function and all the calculation must be done in the radians mode and value of pi should be 3.14

$$f(x) = \cos x - 3x + 1$$

$$f(0) = \cos 0 - 3(0) + 1 = 1 > 0$$

$$f(\pi/2) = \cos(1.57) - 3(1.57) + 1 = 0.0007963 - 4.71 + 1 = -3.7092037 < 0$$

so a real root lies between 0 and  $\pi/2$

$$\text{here } \phi(x) = \frac{1}{3}(\cos x + 1)$$

$$\text{we have } \phi'(x) = -\frac{1}{3}\sin x$$

it is clearly less than 1 as sin is a bounded function and its values lies between -1 and 1

hence iteration method can be applied

let  $x_0 = 0.5$  be the initial approximation then

$$x_1 = \phi(x_0) = \frac{1}{3}[\cos(0.5) + 1] = 0.6258608$$

$$x_2 = \phi(x_1) = \frac{1}{3}[\cos(0.6258608) + 1] = 0.6034863$$

$$x_3 = \phi(x_2) = \frac{1}{3}[\cos(0.6034863) + 1] = 0.6077873$$

$$x_4 = \phi(x_3) = \frac{1}{3}[\cos(0.6077873) + 1] = 0.6069711$$

$$x_5 = \phi(x_4) = \frac{1}{3}[\cos(0.6069711) + 1] = 0.6071264$$

$$x_6 = \phi(x_5) = \frac{1}{3}[\cos(0.6071264) + 1] = 0.6070969$$

$$x_7 = \phi(x_6) = \frac{1}{3} [\cos(0.6070969) + 1] = 0.6071025$$

$$x_8 = \phi(x_7) = \frac{1}{3} [\cos(0.6071025) + 1] = 0.6071014$$

$$x_9 = \phi(x_8) = \frac{1}{3} [\cos(0.6071014) + 1] = 0.6071016$$

$$x_{10} = \phi(x_9) = \frac{1}{3} [\cos(0.6071016) + 1] = 0.6071016$$

## Newton -Raphson Method

This method is one of the most powerful method and well known methods, used for finding a root of  $f(x)=0$  the formula many be derived in many ways the simplest way to derive this formula is by using the first two terms in Taylor's series expansion of the form,

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

setting  $f(x_{n+1}) = 0$  gives,

$$f(x_n) + (x_{n+1} - x_n) f'(x_n) = 0$$

thus on simplification, we get ,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n = 0, 1, 2, \dots$$

## Geometrical interpretation

Let the curve  $f(x)=0$  meet the x-axis at  $x=\alpha$  .it means that  $\alpha$  is the original root of the  $f(x)=0$ . Let  $x_0$  be the point near the root  $\alpha$  of the equation  $f(x)=0$  then the equation of the tangent  $P_0[x_0, f(x_0)]$  is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

This cuts the x-axis at  $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

This is the first approximation to the root  $\alpha$  .if  $P_1[x_1, f(x_1)]$  is the point corresponding to  $x_1$  on the curve then the tangent at  $P_1$  is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

This cuts the x-axis at  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

This is the second approximation to the root  $\alpha$  .Repeating this process we will get the root  $\alpha$  with better approximations quite rapidly.

### Note:

1. When  $f'(x)$  very large .i.e. is when the slope is large, then  $h$  will be small (as assumed) and hence, the root can be calculated in even less time.
2. If we choose the initial approximation  $x_0$  close to the root then we get the root of the equation very quickly.
3. The process will evidently fail if  $f'(x) = 0$  is in the neighborhood of the root. In such cases the regula falsi method should be used.

4. If the initial approximation to the root is not given, choose two say, a and b, such that  $f(a)$  and  $f(b)$  are of opposite signs. If  $|f(a)| < |f(b)|$  then take a as the initial approximation.
5. Newton's Raphson method is also referred to as the method of tangents.

### Example

Find a real root of the equation  $x^3 - x - 1 = 0$  using Newton - Raphson method, correct to four decimal places.

### Solution

$$f(x) = x^3 - x - 1$$

$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$f(2) = 2^3 - 2 - 1 = 8 - 2 - 1 = 5 > 0$$

so the root lies between 1 and 2

$$\text{here } f'(x) = 3x^2 - 1 \quad \text{and} \quad f''(x) = 6x$$

$$f'(1) = 3 \cdot 1^2 - 1 = 2$$

$$f'(2) = 3 \cdot 2^2 - 1 = 11$$

here

$$f''(1) = 6$$

$$f''(2) = 6(2) = 12$$

here  $f(2)$  and  $f''(2)$  have the same signs so  $x_0 = 2$

The second approximation is computed using Newton-Raphson method as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{5}{11} = 1.54545$$

$$f(1.54545) = 1.54545^3 - 1.54541 - 1 = 3.691177 - 1.54541 - 1 = 1.14576$$

$$f'(x) = 3x^2 - 1 = 3(1.54541)^2 - 1 = 3(2.38829) - 1 = 7.16487 - 1 = 6.16487$$

$$x_2 = 1.54545 - \frac{1.14576}{6.16525} = 1.35961$$

$$f(1.35961) = 1.35961^3 - 1.35961 - 1 = 3.691177 - 1.35961 - 1 = 1.14576$$

$$f'(x) = 3x^2 - 1 = 3(1.35961)^2 - 1 = 3(1.84854) - 1 = 5.54562 - 1 = 4.54562$$

$$x_3 = 1.35961 - \frac{1.14576}{4.54562} = 1.05369$$

$$x_3 = 1.35961 - \frac{0.15369}{4.60959 \times 10^{-3}} = 1.32579, \quad f(x_3) = 4.60959 \times 10^{-3}$$

$$x_4 = 1.32579 - \frac{4.60959 \times 10^{-3}}{4.27316} = 1.32471, \quad f(x_4) = -3.39345 \times 10^{-5}$$

$$x_5 = 1.32471 + \frac{3.39345 \times 10^{-5}}{4.26457} = 1.324718, \quad f(x_5) = 1.823 \times 10^{-7}$$

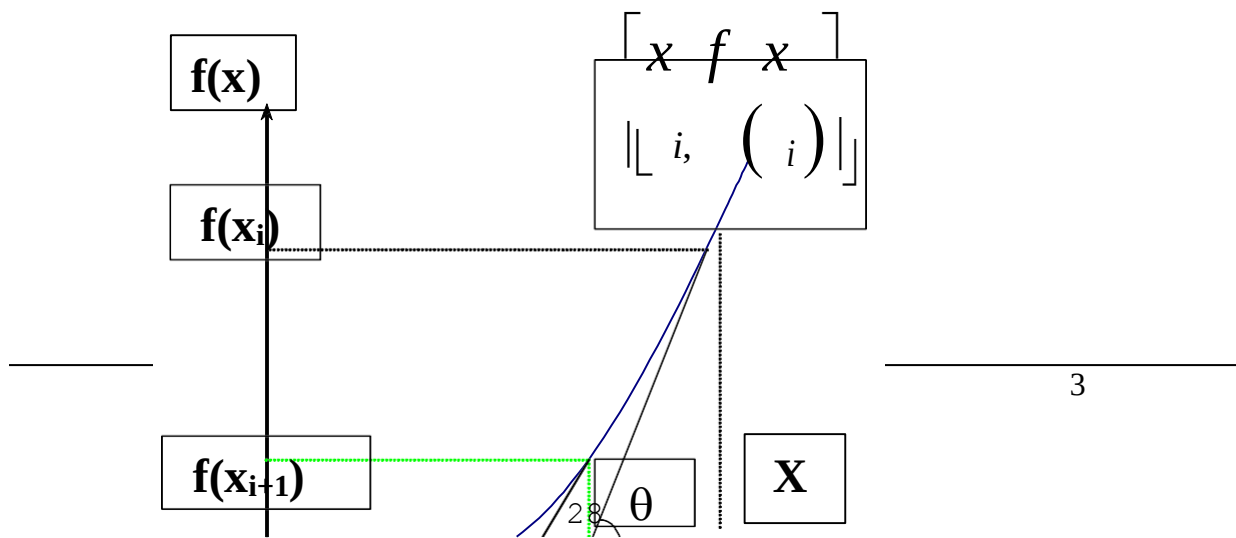
Hence, the required root is 1.3247

### Note!

Methods such as bisection method and the false position method of finding roots of a nonlinear equation  $f(x) = 0$  require bracketing of the root by two guesses. Such methods are called bracketing methods. These methods are always convergent since they are based on reducing the interval between the two guesses to zero in on the root.

In the Newton-Raphson method, the root is not bracketed. Only one initial guess of the root is needed to get the iterative process started to find the root of an equation. Hence, the method falls in the category of open methods.

Newton - Raphson method is based on the principle that if the initial guess of the root of  $f(x) = 0$  is at  $x_i$ , then if one draws the tangent to the curve at  $f(x_i)$ , the point  $x_{i+1}$  where the tangent crosses the x-axis is an improved estimate of the root



### Draw backs of N-R Method

*Divergence at inflection points:*

If the selection of a guess or an iterated value turns out to be close to the inflection point of  $f(x)$ ,  
[where  $f''(x) = 0$ ],  
the roots start to diverge away from the root.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

*Division of zero or near zero:*

If an iteration, we come across the division by zero or a near-zero number, then we get a large magnitude for the next value,  $x_{i+1}$ .

*Root jumping:*

In some case where the function  $f(x)$  is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

*Oscillations near local maximum and minimum:*

Results obtained from N-R method may oscillate about the local max or min without converging on a root but converging on the local max or min. Eventually, it may lead to division to a number close to zero and may diverge.

### Convergence of N-R Method

Let us compare the N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with the general iteration formula

$$x_{n+1} = \phi(x_n),$$

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

The iteration method converges if

$$|\phi'(x)| < 1.$$

Therefore, N-R formula converges, provided

$$|f(x)f''(x)| < |f'(x)|^2$$

in the interval considered.

Newton-Raphson formula therefore converges, provided the initial approximation  $x_0$  is chosen sufficiently close to the root and are continuous and bounded in any small interval containing the root.

### Definition

Let  $x_n = \alpha + \varepsilon_n$ ,  $x_{n+1} = \alpha + \varepsilon_{n+1}$

where  $\alpha$  is a root of  $f(x) = 0$ .

If we can prove that  $\varepsilon_{n+1} = K\varepsilon_n^p$ ,

where  $K$  is a constant and  $\varepsilon_n$  is the error involved at the  $n$ -th step, while finding the root by an iterative method, then the rate of convergence of the method is  $p$ .

The N-R method converges quadratically

$$x_n = \alpha + \varepsilon_n,$$

$$x_{n+1} = \alpha + \varepsilon_{n+1}$$

where  $\alpha$  is a root of  $f(x) = 0$  and  $\varepsilon_n$  is the error involved at the  $n$ -th step, while finding the root by N-R formula

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)} = \frac{\varepsilon_n f'(\alpha + \varepsilon_n) - f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

Using Taylor's expansion, we get

$$\varepsilon_{n+1} = \frac{1}{f'(\alpha) + \varepsilon_n f'(\alpha) + \dots} \left\{ \varepsilon_n [f'(\alpha) + \varepsilon_n f'(\alpha) + \dots] - \left[ f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2} f''(\alpha) + \dots \right] \right\}$$

Since  $\alpha$  is a root,  $f(\alpha) = 0$ . Therefore, the above expression simplifies to

$$\begin{aligned} \varepsilon_{n+1} &= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha) + \varepsilon_n f'(\alpha)} \\ &= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[ 1 + \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)} \right]^{-1} \\ &= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[ 1 - \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)} \right] \\ &= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[ 1 - \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)} \right] \\ &= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[ 1 - \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)} \right] \\ &= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} + O(\varepsilon_n^3) \end{aligned}$$

On neglecting terms of order  $\varepsilon_n^3$  and higher powers, we obtain

$$\varepsilon_{n+1} = K\varepsilon_n^2$$

Where

$$K = \frac{f''(\alpha)}{2f'(\alpha)}$$

It shows that N-R method has second order convergence or converges quadratically.

### Example

Set up Newton's scheme of iteration for finding the square root of a positive number  $N$ .

### Solution

The square root of  $N$  can be carried out as a root of the equation

$$x^2 - N = 0.$$

Let

$$f(x) = x^2 - N.$$

By Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

### In this Problem

$$f(x) = x^2 - N, f'(x) = 2x.$$

Therefore

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2} \left( x_n + \frac{N}{x_n} \right)$$

**Example**

Evaluate  $\sqrt{12}$ , by Newton's formula.

**Solution**

Since

$$\sqrt{9} = 3, \sqrt{16} = 4,$$

We take

$$x_0 = (3+4)/2 = 3.5,$$

$$x_1 = \frac{1}{2} \left( x_0 + \frac{N}{x_0} \right) = \frac{1}{2} \left( 3.5 + \frac{12}{3.5} \right) = 3.4643$$

$$x_2 = \frac{1}{2} \left( 3.4643 + \frac{12}{3.4643} \right) = 3.4641$$

$$x_3 = \frac{1}{2} \left( 3.4641 + \frac{12}{3.4641} \right) = 3.4641$$

Hence

$$\sqrt{12} = 3.4641.$$

Here in this solution we use the iterative scheme developed in the above example and simply replace the previous value in the next iteration and finally come to the result.

**Example**

Find the first three iteration of the equation  $f(x) = x - 0.8 - 0.2 \sin x$  in the interval  $[0, \pi/2]$ .

**Solution**

$$f(0) = 0 - 0.8 - 0.2 \sin(0) = 0 - 0.8 - 0.2(0) = -0.8$$

$$\begin{aligned} f(1.57) &= 1.57 - 0.8 - 0.2 \sin(1.75) \\ &= 1.57 - 0.8 - 0.2(0.99999) \\ &= 1.57 - 0.8 - 0.199998 = 0.570002 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(0) = 1 - 0.2 \cos(0) = 1 - 0.2 = 0.8$$

here  $|f(0)|$  is greater than  $x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-0.8}{0.8} = 1$$

now

$$\begin{aligned} f(1) &= 1 - 0.8 - 0.2 \sin(1) \\ &= 1 - 0.8 - 0.1683 \\ &= 0.0317 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(1) = 1 - 0.2 \cos(1) = 1 - 0.1081 = 0.8919$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{0.0317}{0.8919} = 1 - 0.0355 = 0.9645$$

$$\begin{aligned} f(0.9645) &= 0.9645 - 0.8 - 0.2 \sin(0.9645) \\ &= 0.9645 - 0.8 - 0.1645 \\ &= 0.0002 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(0.9645) = 1 - 0.2 \cos(0.9645) = 1 - 0.11396 = 0.88604$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.9645 - \frac{0.0002}{0.88604} = 0.9645 - 0.00022 = 0.9643$$

NOTE: In the above question the function is a transcendental function and we have to perform all the calculation in the radians mode and the value of pi should be taken as 3.14

### Example

Consider the equation  $f(x) = 4x \cos x - (x - 2)^2$  find the root of the equation in the range  $0 \leq x \leq 8$

### Solution

$$f(x) = 4x \cos x - (x-2)^2$$

here

$$f(0) = 4(0) \cos 2(0) - (0-2)^2 = -4$$

$$\begin{aligned} f(8) &= 4(8) \cos 2(8) - (8-2)^2 \\ &= 32 \cos 16 - (6)^2 = -30.6451 - 36 = -66.6451 \end{aligned}$$

$$\begin{aligned} f'(x) &= 4 \cos 2x - 8x \sin 2x - 2(x-2) \\ &= 4 \cos 2x - 8x \sin 2x - 2(x-2) \end{aligned}$$

$$\begin{aligned} f'(8) &= 4 \cos 16 - 64 \sin 16 - 2(8-2) \\ &= -3.8306 + 18.4250 - 12 = 2.5952 \end{aligned}$$

since  $|f(8)|$  is greater so  $x_0 = 8$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 8 - \frac{(-66.6451)}{2.5952} = 33.6801$$

$$\begin{aligned} f(33.6801) &= 4(33.6801) \cos 2(33.6801) - (33.6801-2)^2 \\ &= -24.6545 - 1003.6 = -1028.25 \end{aligned}$$

$$\begin{aligned} f'(33.6801) &= 4 \cos 2(33.6801) - 8(33.6801) \sin 2(33.6801) - 2(33.6801-2) \\ &= -0.7320 + 264.89 = -63.3602 \end{aligned}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 33.6801 + \frac{1028.25}{200.79} = 38.8011$$

$$\begin{aligned} f(38.8011) &= 4(38.8011) \cos 2(38.8011) - (38.8011-2)^2 \\ &= -91.8361 - 1354.3 = 1446.14 \end{aligned}$$

$$\begin{aligned} f'(38.8011) &= 4 \cos 2(38.8011) - 8(38.8011) \sin 2(38.8011) - 2(38.8011-2) \\ &= -2.3668 - 250.236 - 73.6022 = -326.205 \end{aligned}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 38.8011 + \frac{1446.14}{-326.205} = 38.8011 + 4.4332 = 43.2343$$

### Example

Perform three iteration of the equation  $\ln(x-1) + \cos(x-1) = 0$  when  $1.2 \leq x \leq 2$ . Use Newton Raphson method to calculate the root.

### Solution

Here

$$\ln(x-1) + \cos(x-1) = 0 \text{ when } 1.2 \leq x \leq 2$$

$$f(x) = \ln(x-1) + \cos(x-1)$$

$$f(x) = \ln(x-1) + \cos(x-1)$$

$$\begin{aligned} f(1.2) &= \ln(1.2-1) + \cos(1.2-1) \\ &= -1.6094 + 0.9801 = -0.6293 \end{aligned}$$

$$\begin{aligned} f(2) &= \ln(2-1) + \cos(2-1) \\ &= 0 + 0.5403 = 0.5403 \end{aligned}$$

$$\text{now } f(x) = \ln(x-1) + \cos(x-1)$$

$$f'(x) = \frac{1}{x-1} - \sin(x-1)$$

$$\begin{aligned} f'(1.2) &= \frac{1}{1.2-1} - \sin(1.2-1) \\ &= 5 - 0.1986 = 4.8014 \end{aligned}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{-0.6293}{4.8014} = 1.2 + 0.1311 = 1.3311$$

$$\begin{aligned} f(1.3311) &= \ln(1.3311-1) + \cos(1.3311-1) \\ &= -1.1053 + 0.9457 = -0.1596 \end{aligned}$$

$$\begin{aligned} f'(1.3311) &= \frac{1}{1.3311-1} - \sin(1.3311-1) \\ &= 3.0202 - 0.3251 = 2.6951 \end{aligned}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3311 - \frac{-0.1596}{2.6951} = 1.3311 + 0.0592 = 1.3903$$

$$\begin{aligned} f(1.3903) &= \ln(1.3903-1) + \cos(1.3903-1) \\ &= -0.9408 + 0.9248 = -0.016 \end{aligned}$$

$$\begin{aligned} f'(1.3903) &= \frac{1}{1.3903-1} - \sin(1.3903-1) \\ &= 2.5621 - 0.3805 = 2.1816 \end{aligned}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.3903 - \frac{-0.016}{2.1816} = 1.3903 + 0.0073 = 1.3976$$

### Secant Method

The secant method is modified form of Newton-Raphson method. If in Newton-Raphson method; we replace the derivative  $f'(x_n)$  by the difference ratio, i.e.,

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Where  $x_n$  and  $x_{n-1}$  are two approximations of the root we get

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_n f(x_n) - x_n f(x_{n-1}) - f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \end{aligned}$$

Provided  $f(x_n) \neq f(x_{n-1})$

This method requires two starting values  $x_0$  and  $x_1$ ; values at both these points are calculated which gives two points on the curve the next point on the curve are obtained by using the derived formula. We continue this procedure until we get the root of required accuracy.

### Geometrical Interpretation

Geometrically, the secant method corresponds to drawing secants rather than tangents to obtain various approximations to root  $\alpha$ ;

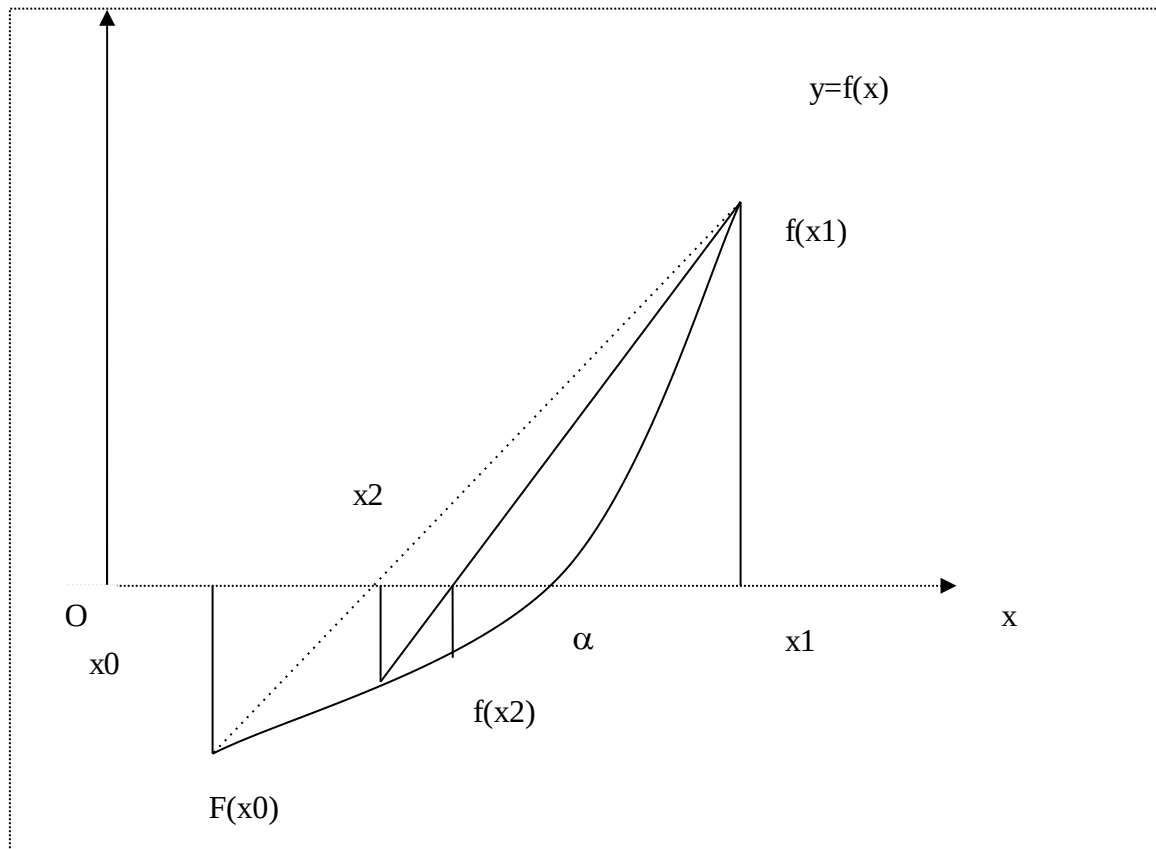
To obtain  $x_2$  we find the intersection between the secant through the points  $(x_0, f(x_0))$

And  $(x_1, f(x_1))$  and the x-axis.

It is experienced that the secant method converges rather quickly. There is a possibility of the divergence if the two roots lie on the same side of the curve. The order of the

convergence of the secant method equal to  $\frac{(1+\sqrt{5})}{2} = 1.618$ , which shows that this

method has the order of convergence slightly inferior than that of Newton-Raphson method. In this method  $f(x)$  is not required to change signs between the estimate.



### Some more description

We choose  $x_0, x_1$  close to the root 'a' of  $f(x) = 0$  such that  $f(x_0) \neq f(x_1)$

As a next approximation  $x_2$  is obtained as the point of intersection of  $y = 0$  and the chord passing through the points

$$(x_0, f(x_0)), (x_1, f(x_1)).$$

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0),$$

Putting  $y = 0$  we get

$$x_2 = x = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

### Convergence of Secant Method

Here the sequence is generated by the rule

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Starting with  $x_0$  and  $x_1$  as  $\{x_0, x_1, \dots\}$

It will converge to 'a', that is  $f(a) = 0$

### NOTE

The Secant method converges faster than linear and slower than Newton's quadratic

**Example**

Do three iterations of secant method to find the root of

$$f(x) = x^3 - 3x + 1 = 0,$$

Taking

$$x_0 = 1, x_1 = 0.5$$

$$n = 1,$$

$$f(x_0) = f(1) = 1^3 - 3(1) + 1 = -1$$

$$f(x_1) = f(0.5) = 0.5^3 - 3(0.5) + 1 = 0.125 - 1.5 + 1 = -0.375$$

$$\begin{aligned} x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ &= \frac{(1)(-0.375) - (0.5)(-1)}{-0.375 - (-1)} = 0.2 \end{aligned}$$

$$n = 2,$$

$$f(x_2) = f(0.2) = 0.2^3 - 3(0.2) + 1 = 0.008 - 0.6 + 1 = 0.408$$

$$\begin{aligned} x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\ &= \frac{(0.5)(0.408) - 0.2(-0.375)}{0.408 - (-0.375)} = 0.3563 \end{aligned}$$

$$n = 3,$$

$$f(x_3) = f(0.3563) = 0.3563^3 - 3(0.3563) + 1 = 0.04523 - 1.0689 + 1 = -0.02367$$

$$\begin{aligned} x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\ &= \frac{(0.2) f(0.3563) - 0.3563 f(0.2)}{f(0.3563) - f(0.2)} \end{aligned}$$

$$x_5 = 0.3473, f(x_5) = -0.0000096$$

$$\text{and } |x_5 - x_4| = 0.0004.$$

Though  $x_5$  is not the true root, yet this is good approximation to the root and convergence is faster than bisection.

**Example**

Find the root of  $2 \cosh x \sin x = 1$  using the secant method, with accuracy of 4 decimal point. Take 0.4 and 0.5 as the two starting values

$$f(x) = 2 \cosh x \sin x - 1$$

$$\begin{aligned} \text{now } f(x_0) &= 2 \cosh x_0 \sin x_0 - 1 \\ &= 2 \cosh 0.4 \sin 0.4 - 1 \\ &= 2 \times 1.081 \times 0.3894 - 1 = -0.1580 \end{aligned}$$

$$\begin{aligned} f(x_1) &= 2 \cosh x_1 \sin x_1 - 1 \\ &= 2 \cosh 0.5 \sin 0.5 - 1 \\ &= 2 \times 1.1276 \times 0.4794 - 1 = 0.0811 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ &= \frac{0.4 \times 0.0811 - 0.5 \times -0.1580}{0.0811 - (-0.1580)} = \frac{0.03244 + 0.0790}{0.2391} = 0.4661 \end{aligned}$$

$$\begin{aligned} f(x_2) &= 2 \cosh x_2 \sin x_2 - 1 \\ &= 2 \cosh 0.5 \sin 0.5 - 1 \\ &= 2 \times 1.1106 \times 0.4494 - 1 = -0.0811 \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\ &= \frac{0.5 \times -0.0811 - 0.4661 \times 0.0811}{-0.0811 - 0.0811} = \frac{0.009 - 0.0378}{-0.1622} = 0.4668 \end{aligned}$$

$$\begin{aligned} f(x_3) &= 2 \cosh x_3 \sin x_3 - 1 \\ &= -0.00009 \end{aligned}$$

$$\begin{aligned} x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\ &= \frac{0.4661 \times -0.00009 - 0.4668 \times -0.0811}{-0.00009 - (-0.0811)} = \frac{-0.000042 + 0.0378}{-0.00009 + 0.0811} = 0.4667 \end{aligned}$$

### Comparison:

In secant method, we do not check whether the root lies in between two successive approximates  $X_{n-1}$ , and  $X_n$ .

This checking was imposed after each iteration, in Regula –Falsi method.

### Muller's Method

In *Muller's method*,  $f(x) = 0$  is approximated by a second degree polynomial; that is by a quadratic equation that fits through three points in the vicinity of a root. The roots of this quadratic equation are then approximated to the roots of the equation  $f(x) = 0$ . This method is iterative in nature and does not require the evaluation of derivatives as in Newton-

Raphson method. This method can also be used to determine both real and complex roots of  $f(x) = 0$ .

Suppose,  $x_{i-2}, x_{i-1}, x_i$

be any three distinct approximations to a root

Of  $f(x) = 0$ .

$$f(x_{i-2}) = f_{i-2}, f(x_{i-1}) = f_{i-1}$$

$$f(x_i) = f_i.$$

Noting that any three distinct points in the  $(x, y)$ -plane uniquely; determine a polynomial of second degree.

A general polynomial of second degree is given by

Noting that any three distinct points in the  $(x, y)$ -plane uniquely; determine a polynomial of second degree.

A general polynomial of second degree is given by

$$f(x) = ax^2 + bx + c$$

Suppose, it passes through the points

$$(x_{i-2}, f_{i-2}), (x_{i-1}, f_{i-1}), (x_i, f_i)$$

Then the following equations will be satisfied

$$ax_{i-2}^2 + bx_{i-2} + c = f_{i-2}$$

$$ax_{i-1}^2 + bx_{i-1} + c = f_{i-1}$$

$$ax_i^2 + bx_i + c = f_i$$

Eliminating  $a, b, c$ , we obtain

$$\begin{vmatrix} x_{i-2}^2 & x_{i-2} & 1 & f_{i-2} \\ x_{i-1}^2 & x_{i-1} & 1 & f_{i-1} \\ x_i^2 & x_i & 1 & f_i \end{vmatrix} = 0$$

This can be written as

$$f = \frac{(x - x_{i-1})(x - x_i)}{(x_{i-2} - x_{i-1})(x_{i-2} - x_i)} f_{i-2} + \frac{(x - x_{i-2})(x - x_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)} f_{i-1} + \frac{(x - x_{i-2})(x - x_{i-1})}{(x_i - x_{i-2})(x_i - x_{i-1})} f_i$$

We further define

$$\lambda = \frac{h}{h_i} = \frac{x - x_i}{x_i - x_{i-1}}$$

$$\lambda_i = \frac{h_i}{h_{i-1}}$$

$$\delta_i = 1 + \lambda_i$$

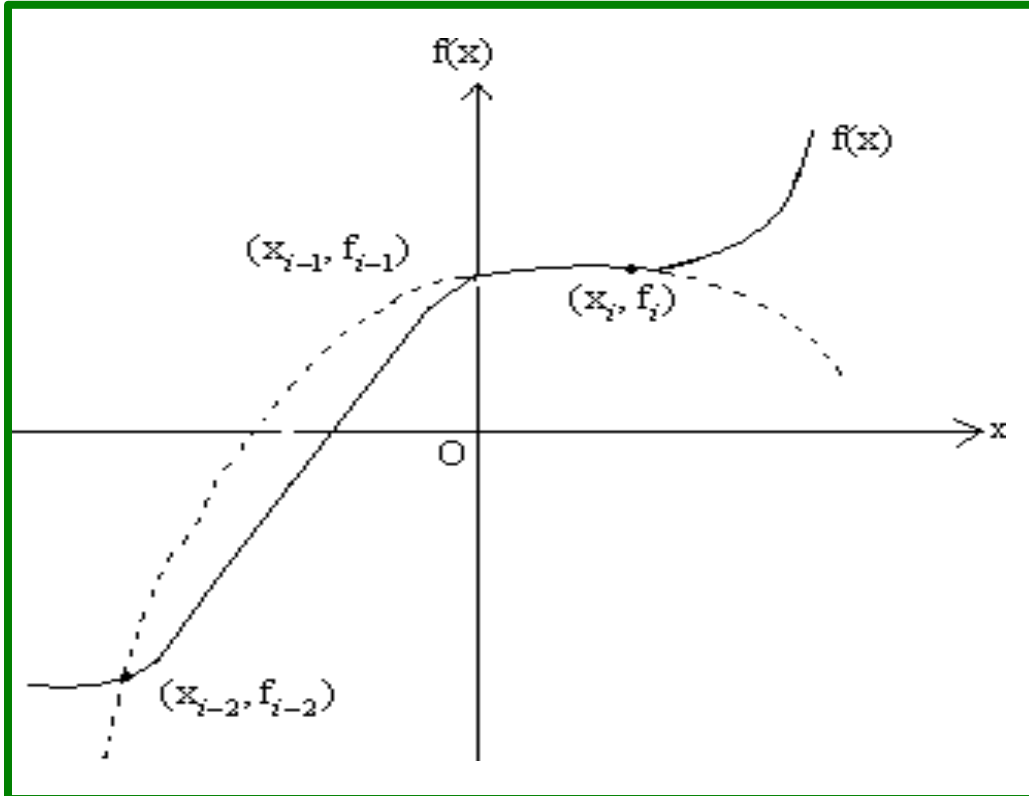


**Fig: Quadratic polynomial**

With these substitutions we get a simplified Equation as

$$f = \frac{1}{\delta_i} [\lambda(\lambda+1)\lambda^2 f_{i-2} - \lambda(\lambda+1+\lambda^{-1})\lambda\delta f_{i-1} + (\lambda+1+\lambda_i^{-1})\lambda_i f_i]$$

Or

$$f = \lambda^2 (f_{i-2} \lambda^2 - f_{i-1} \lambda \delta + f_i \lambda) \delta^{-1} \\ + \lambda [f_{i-2} \lambda^2 - f_{i-1} \delta^2 \\ + f_i (\lambda_i + \delta_i)] \delta_i^{-1} + f_i$$

To compute  $\lambda$ , set  $f = 0$ , we obtain  
 $\lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i) \lambda^2 + g_i \lambda + \delta_i f_i = 0$

Where  
 $g_i = f_{i-2} \lambda^2 - f_{i-1} \delta^2 + f_i (\lambda_i + \delta_i)$

A direct solution will lead to loss of accuracy and therefore to obtain max accuracy we rewrite as:

$$\frac{f_i \delta_i}{\lambda^2} + \frac{g_i}{\lambda} + \lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i) = 0$$

So that,

$$\frac{1}{\lambda} = \frac{-g_i \pm [g_i^2 - 4 f_i \delta_i \lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i)]^{1/2}}{2 f_i \delta_i}$$

Or

$$\lambda = \frac{-2 f_i \delta_i}{g_i \pm [g_i^2 - 4 f_i \delta_i \lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i)]^{1/2}}$$

Here, the positive sign must be so chosen that the denominator becomes largest in magnitude.

We can get a better approximation to the root, by using

$$x_{i+1} = x_i + h_i \lambda$$

## Muller's Method

Muller's method is a generalization of the secant method, in the sense that it does not require the derivative of the function. It is an iterative method that requires three starting points  $(p_0, f(p_0))$ ,  $(p_1, f(p_1))$ , and  $(p_2, f(p_2))$ . A parabola is constructed that passes through the three points; then the quadratic formula is used to find a root of the quadratic for the next approximation. It has been proved that near a simple root Muller's method converges faster than the secant method and almost as fast as Newton's method. The method can be used to find real or complex zeros of a function and can be programmed to use complex arithmetic.

### Example

Do two iterations of Muller's method to solve  $x^3 - 3x + 1 = 0$  starting with  $x_0 = 0.5$ ,  $x_1 = 0$ ,  $x_2 = 1$

### Solution

$$f(x_0) = f_0 = (0.5)^3 - 3(0.5) + 1 = -0.375$$

$$f(x_1) = f_1 = (1)^3 - 3(1) + 1 = -1$$

$$f(x_2) = f_2 = 0 - 3 \times 0 + 1 = +1$$

$$c = f_0 = -0.375$$

$$h_1 = x_1 - x_0 = 0.5$$

$$h_2 = x_2 - x_1 = 0.5$$

$$a = \frac{h_1 f_2 - (h_1 + h_2) f_1 + h_2 f_0}{h_1 h_2 (h_1 + h_2)}$$

$$= \frac{(0.5)(-1) - (-0.375) + (0.5)}{0.25} = 1.5$$

$$b = \frac{f_1 - f_0 - ah_1^2}{h_1} = -2$$

$$\therefore x = x_0 - \frac{2c}{+b - \sqrt{b^2 - 4ac}}$$

$$= 0.5 - \frac{2(-0.375)}{-2 - \sqrt{4 + 4(1.5)(0.375)}}$$

$$= 0.5 - \frac{-0.75}{-2 - \sqrt{4 + 2.25}} = 0.33333 < 0.5$$

Take

$$x_2 = 0, x_0 = 0.33333, x_1 = 0.5$$

$$h_1 = x_1 - x_0 = 0.16667, h_2 = x_0 - x_2 = 0.33333$$

$$c = f_0 = f(0.33333)$$

$$= x_0^3 - 3x_0 + 1 = 0.037046$$

$$f_1 = x_1^3 - 3x_1 + 1 = -0.375$$

$$f_2 = x_2^3 - 3x_2 + 1 = 1$$

$$a = \frac{h_2 f_1 - (h_1 + h_2) f_0 + h_1 f_2}{h_1 h_2 (h_1 + h_2)} = \frac{0.023148}{0.027778}$$

$$= 0.8333$$

$$b = \frac{f_1 - f_0 - ah_1^2}{h_1} = -2.6$$

$$x = x_0 - \frac{2c}{b - \sqrt{b^2 - 4ac}}$$

$$= 0.333333 - \frac{0.074092}{-5.2236}$$

$$= 0.3475 > 0.33333 = x_0$$

For third iteration take,

$$x_2 = 0.333333, \quad x_0 = 0.3475, \quad x_1 = 0.5$$

### Graeffe's Root Squaring Method

GRAEFFE'S ROOT SQUARING METHOD is particularly attractive for finding all the roots of a polynomial equation. Consider a polynomial of third degree  $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$f(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3$$

$$f(x)f(-x) = a_3^2 x^6 - (a_2^2 - 2a_1 a_3) x^4$$

$$+ (a_1^2 - 2a_0 a_2) x^2 - a_0^2$$

$$f(x)f(-x) = a_3^2 t^3 - (a_2^2 - 2a_1 a_3) t^2$$

$$+ (a_1^2 - 2a_0 a_2) t - a_0^2$$

The roots of this equation are squares or  $2i$  ( $i = 1$ ), powers of the original roots. Here  $i = 1$  indicates that squaring is done once.

The same equation can again be squared and this squaring process is repeated as many times as required. After each squaring, the coefficients become large and overflow is possible as  $i$  increase.

Suppose, we have squared the given polynomial ' $i$ ' times, then we can estimate the value of the roots by evaluating  $2i$  root of

$$\left| \frac{a_i}{a_{i-1}} \right|, \quad i = 1, 2, \dots, n$$

Where  $n$  is the degree of the given polynomial.

The proper sign of each root can be determined by recalling the original equation. This method fails, when the roots of the given polynomial are repeated.

This method has a great advantage over other methods in that it does not require prior information about approximate values etc. of the roots. But it is applicable to polynomials only and it is capable of giving all the roots. Let us see the case of the polynomial equation having real and distinct roots. Consider the following polynomial equation

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n \quad (1)$$

separating the even and odd powers of  $x$  and squaring we get

$$(x^n + a_2 x^{n-2} + a_4 x^{n-4} + \dots)^2 = (a_1 x^{n-1} + a_3 x^{n-3} + a_5 x^{n-5} + \dots)^2$$

putting  $x^2 = y$  and simplifying we get the new equation

$$y^n + b_1 y^{n-1} + b_2 y^{n-2} + \dots + b_{n-1} y + b_n = 0 \quad (2)$$

$$b_1 = -a_1^2 + 2a_2$$

$$b_2 = a_1^2 - 2a_1 a_3 + 2a_4$$

$$\dots \dots \dots$$

$$b_n = (-1)^n a_n^2$$

if  $p_1, p_2, \dots, p_n$  be the roots of eq 1 then the roots of the equation 2 are  $p_1^2, p_2^2, \dots, p_n^2$

### Example

Using Graeffe root squaring method, find all the roots of the equation  $x^3 - 6x^2 + 11x - 6 = 0$

#### Solution

Using Graeffe root squaring method, the first three squared polynomials are as under:

For  $i = 1$ , the polynomial is

$$\begin{aligned} & x^3 - (36 - 22)x^2 + (121 - 72)x - 36 \\ & = x^3 - 14x^2 + 49x - 36 \end{aligned}$$

For  $i = 2$ , the polynomial is

$$x^3 - (196 - 98)x^2 + (2401 - 1008)x - 1296$$

$$= x^3 - 98x^2 + 1393x - 1296$$

For  $i = 3$ , the polynomial is

$$x^3 - (9604 - 2786)x^2 + (1940449 - 254016)x - 1679616$$

$$= x^3 - 6818x^2 + 16864333x - 1679616$$

The roots of polynomial are

$$\sqrt{\frac{36}{49}} = 0.85714, \quad \sqrt{\frac{49}{14}} = 1.8708, \quad \sqrt{\frac{14}{1}} = 3.7417$$

Similarly the roots of the polynomial 2 are

$$\sqrt[4]{\frac{1296}{1393}} = 0.9821, \quad \sqrt[4]{\frac{1393}{98}} = 1.9417, \quad \sqrt[4]{\frac{98}{1}} = 3.1464$$

Still better estimates of the roots obtained from polynomial (3) are

$$\sqrt[8]{\frac{1679616}{1686433}} = 0.99949, \quad \sqrt[8]{\frac{1686433}{6818}} = 1.99143, \quad \sqrt[8]{\frac{6818}{1}} = 3.0144$$

The exact values of the roots of the given polynomial are 1, 2 and 3.

### Example

Apply Graeffe's root squaring method to solve the equation  $x^3 - 8x^2 + 17x - 10 = 0$

### Solution

$$f(x) = x^3 - 8x^2 + 17x - 10$$

here three changes are observed from +ve to -ve, -ve to +ve, +ve to -ve

hence according to deccarte rule of signs  $f(x)$  may have three positive roots

rewriting eq as  $x(x^2 + 17) = (8x^2 + 10)$

and squaring on both sides and putting  $x^2 = y$

$$y(y + 17)^2 = (8y + 10)^2$$

$$y^3 + 34y^2 + 289 = 64y^2 + 160y + 100$$

$$y(y^2 + 129) = 30y^2 + 100$$

putting  $y^2 = z$  we get

$$z(z + 129)^2 = (30z + 100)^2$$

$$z^3 + 258z^2 + 16641z = 900z^2 + 6000z + 10000$$

$$z(z^2 + 16641) = (642z^2 + 10000)$$

squaring again and putting  $z^2 = u$  we get

$$u(u + 16641)^2 = (642u + 10000)^2$$

$$u^3 + 33282u^2 + 276922881u = 412164u^2 + 12840000u + 10^8$$

$$u^3 - 378882u^2 + 264082u - 10^8 = 0$$

if the roots of the eq1 are  $p_1, p_2, p_3$ , and those of equation 5 are  $q_1, q_2, q_3$

$$p_1 = (q_1)^{1/8} = (-\lambda_1)^{1/8} = (378882)^{1/8} = 4.9809593 \cong 5$$

$$p_2 = (q_2)^{1/8} = (-\lambda_2 / \lambda_1)^{1/8} = (264082 / 378882)^{1/8} = 0.9558821 \cong 1$$

$$p_3 = (q_3)^{1/8} = (-\lambda_3 / \lambda_2)^{1/8} = (10^8 / 264082)^{1/8} = 2.1003064 \cong 2$$

$$\text{here } f(5) = f(2) = f(1) = 0$$

hence all are the roots of the equation

### Example

Find all the root of the equation  $x^4 - 3x + 1 = 0$

### Solution

$$f(x) = x^4 - 3x + 1 \quad (1)$$

here there are two changes in the sign so there are two positive roots of the equation and there are no change in the roots of the equation  $f(-x)$  so two positive real roots and two are complex roots

rewriting the above equation as

$$x^4 + 1 = 3x$$

squaring the equation and putting  $x^2 = y$

we have

$$(y^2 + 1)^2 = 9y$$

squaring again and putting  $y^2 = z$

$$(z + 1)^4 = 81z$$

$$z^4 + 4z^3 + 6z^2 - 77z + 1 = 0$$

$$z^4 + 6z^2 + 1 = -z(4z^2 - 77) \quad (2)$$

squaring once again and putting  $z^2 = u$ , we get

$$(u^2 + 6u + 1) = u(4u - 77)^2$$

$$u^4 - 4u^3 + 645u^2 - 5917u + 1 = 0 \quad (3)$$

if  $p_1, p_2, p_3, p_4$  are the roots of the equation 1 and  $q_1, q_2, q_3, q_4$  are roots of equation 3 then

$$p_1 = (q_1)^{1/8} = (-\lambda_1)^{1/8} = (4)^{1/8} = 1.189207$$

$$p_2 = (q_2)^{1/8} = \left[ \frac{-\lambda_2}{\lambda_1} \right]^{1/8} = \left[ \frac{654}{4} \right]^{1/8} = 1.8909921$$

$$p_3 = (q_3)^{1/8} = \left[ \frac{-\lambda_3}{\lambda_2} \right]^{1/8} = \left[ \frac{5917}{654} \right]^{1/8} = 1.3169384$$

$$p_4 = (q_4)^{1/8} = \left[ \frac{-\lambda_4}{\lambda_3} \right]^{1/8} = \left[ \frac{1}{5917} \right]^{1/8} = 0.3376659$$

from equation (2) and (3)

from equation 2 and 3, we observe the magnitude of the coefficients  $\lambda_1$  and  $\lambda_2$  have become constant which implies  $p_1$  and  $p_4$  are the real roots, then  $p_2$  and  $p_3$  are real roots,

let the complex roots be  $\rho_2 e^{\pm i\varphi_2} = \xi_2 + i\eta_2$

from equation (3) it's magnitude is given by

$$\rho_2^{2(2^3)} = \frac{\lambda_3}{\lambda_1} = \frac{5917}{4} \quad \therefore \rho_2 = 1.5780749$$

also from equation 1 sum of roots is zero i.e  $p_1 + 2\xi_2 + p_4 = 0$

$$\xi_2 = -1/2(p_1 + p_4) = -0.7634365$$

$$\eta_2 = \sqrt{\rho_2^2 - \xi_2^2} = \sqrt{1.9074851} = 1.3811173$$

hence, the four roots are 1.1892071, 0.3376659, -0.734365 and  $\pm 1.3811173i$

### Revision Example

Obtain the Newton-Raphson extended formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \frac{[f(x_0)]^2}{[f'(x_0)]^2} f''(x_0)$$

For finding the root of the equation  $f(x)=0$

**Solution**

Expanding  $f(x)$  by Taylor's series and retaining up to second order term,

$$0 = f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0)$$

Therefore,

$$f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{(x_1 - x_0)^2}{2} f''(x_0) = 0$$

This can also be written as

$$f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2} \frac{[f(x_0)]^2}{[f'(x_0)]^2} f''(x_0) = 0$$

Thus, the Newton-Raphson extended formula is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0)$$

This is also known as Chebyshev's formula of third order

## Solution of Linear System of Equations and Matrix Inversion

The solution of the system of equations gives  $n$  unknown values  $x_1, x_2, \dots, x_n$ , which satisfy the system simultaneously. If  $m > n$ , the system usually will have an infinite number of solutions.

If  $|A| \neq 0$  then the system will have a unique solution.

If  $|A| = 0$ ,

Then there exists no solution.

Numerical methods for finding the solution of the system of equations are classified as direct and iterative methods. In direct methods, we get the solution of the system after performing all the steps involved in the procedure. The direct methods consist of elimination methods and decomposition methods.

Under elimination methods, we consider, Gaussian elimination and Gauss-Jordan elimination methods

Crout's reduction also known as Cholesky's reduction is considered under decomposition methods.

Under iterative methods, the initial approximate solution is assumed to be known and is improved towards the exact solution in an iterative way. We consider Jacobi, Gauss-Seidel and relaxation methods under iterative methods.

## Gaussian Elimination Method

In this method, the solution to the system of equations is obtained in two stages.

- i) the given system of equations is reduced to an equivalent upper triangular form using elementary transformations
- ii) the upper triangular system is solved using back substitution procedure

This method is explained by considering a system of  $n$  equations in  $n$  unknowns in the form as follows

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array} \right\}$$

Stage I: Substituting the value of  $x_1$  from first equation into the rest

$$\left. \begin{array}{l} x_1 + a'_{12}x_2 + \dots + a'_{1n}x_n = b'_1 \\ a'_{22}x_2 + \dots + a'_{2n}x_n = b'_2 \\ \vdots \\ a'_{n2}x_2 + \dots + a'_{nn}x_n = b'_n \end{array} \right\}$$

Now, the last  $(n - 1)$  equations are independent of  $x_1$ , that is,  $x_1$  is eliminated from the last  $(n - 1)$  equations.

This procedure is repeated and  $x_2$  is eliminated from 3rd, 4th, ...,  $n$ -th equations

The same procedure is repeated till the given system assumes the upper triangular form:

$$\left. \begin{array}{l} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n = d_1 \\ c_{22}x_2 + \cdots + c_{2n}x_n = d_2 \\ \vdots \\ c_{nn}x_n = d_n \end{array} \right\}$$

Stage II: The values of the unknowns are determined by backward substitution.

First  $x_n$  is found from the last equation and then substitution this value of  $x_n$  in the preceding equation will give the value of  $x_{n-1}$ . Continuing this way, we can find the values of all other unknowns

### Example

Solve the following system of equations using Gaussian elimination method

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

### Solution

Stage I (*Reduction to upper-triangular form*):

Divide first equation by 2 and then subtract the resulting equation (multiplied by 4 and  $-2$ ) from the 2nd and 3rd equations respectively. Thus, we eliminate  $x$  from the 2nd and 3rd equations.

The resulting new system is given by

$$\left. \begin{array}{l} x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\ -2y - z = -7 \\ 6y - 2z = 6 \end{array} \right\}$$

Now, we divide the 2nd equation by  $-2$  and eliminate  $y$  from the last equation and the modified system is given by

$$\left\{ \begin{array}{l} x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\ y + \frac{z}{2} = \frac{7}{2} \\ -5z = -15 \end{array} \right\}$$

Stage II (Backward substitution):

From the last equation, we get

$$z = 3$$

Using this value of  $z$ , the second equation gives

$$y = \frac{7}{2} - \frac{3}{2} = 2$$

Putting these values of  $y$  and  $z$  in the first equation, we get

$$x = \frac{5}{2} + \frac{3}{2} - 3 = 1$$

Thus, the solution of the given system is given by

$$x = 1, y = 2, z = 3$$

### Partial and Full Pivoting

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order of equations is called pivoting.

### Partial pivoting

If the pivot happens to be zero, then the  $i$ -th column elements are searched for the numerically largest element. Let the  $j$ -th row ( $j > i$ ) contains this element, then we interchange the  $i$ -th equation with the  $j$ -th equation and proceed for elimination. This process is continued whenever pivots become zero during elimination.

For example, let us examine the solution of the following simple system

$$10^{-5}x_1 + x_2 = 1$$

$$x_1 + x_2 = 2$$

Using Gaussian elimination method with and without partial pivoting, assuming that we require the solution accurate to only four decimal places. The solution by Gaussian elimination gives  $x_1 = 0, x_2 = 1$ .

If we use partial pivoting, the system takes the form

$$\begin{aligned}x_1 + x_2 &= 2 \\ 10^{-5}x_1 + x_2 &= 1\end{aligned}$$

Using Gaussian elimination method, the solution is found to be  $x_1 = 1$ ,  $x_2 = 1$ , which is a meaningful and perfect result.

In full pivoting (or complete pivoting), we interchange rows and columns, such that the largest element in the matrix of the variables also get changed. Full pivoting, in fact, is more complicated than the partial pivoting. Partial pivoting is preferred for hand computation.

The general form of a system of  $m$  linear equations in  $n$  unknowns  $x_1, x_2, x_3, \dots, x_n$  can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Using matrix notation, the above system can be written in compact form as  
 $[A](X) = (B)$

**Note:**

1. This method fails if any of the pivots become zero in such cases, by interchanging the rows we can get the non-zero pivots.

### Example

Solve the system of equations by Gaussian elimination method with partial pivoting.

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

### Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

To start with, we observe that the pivot element  $a_{11} = 1 (\neq 0)$ .

However, a glance at the first column reveals that the numerically largest element is 3 which is in second row. Hence  $R_1 \leftrightarrow R_2$

Thus the given equation takes the form after partial pivoting

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \\ 16 \end{bmatrix}$$

Stage I (Reduction to upper triangular form):

$$\begin{bmatrix} 3 & 3 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$$

Stage II (Back substitution):

$$z = 3$$

$$-y + 1 = 0 \text{ or } y = 1$$

$$x + 1 + 4 = 8 \text{ or } x = 3$$

### Example

Solve the following system of equations by Gaussian elimination method with partial pivoting

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 6.5x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

### Solution

In matrix notation, the given system can be written as

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 26 \\ 21 \end{bmatrix}$$

To start with, we observe that the pivot row, that is, the first row has a zero pivot element ( $a_{11} = 0$ ). This row should be interchanged with any row following it, which on becoming a pivot row should not have a zero pivot element.

While interchanging rows it is better to interchange the first and fourth rows, which is called partial pivoting and get,

$$\begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix}$$

Stage I (Reduction to upper-triangular form):

$$\begin{bmatrix} 1 & 4 & 4 & 0 \\ 0 & 8.2222 & 3.2222 & 4 \\ 0 & 3.2222 & 4.7222 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ x_2^1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.333 \\ 22.6666 \\ 16.6666 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 4 & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 3.4594 & 0.4324 \\ 0 & 0 & 0.4324 & 6.0540 \end{bmatrix} \begin{bmatrix} x \\ x_2^1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.333 \\ 2.7568 \\ 7.78836 \\ 12.9728 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 4 & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 1 & 0.1250 \\ 0 & 0 & 0 & 5.999 \end{bmatrix} \begin{bmatrix} x \\ x_2^1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 2.7568 \\ 2.2500 \\ 11.9999 \end{bmatrix}$$

Stage II

Back substitution

$$x_1 = 1.0, \quad x_2 = 1.0, \quad x_3 = 2.0, \quad x_4 = 2.0$$

**Example**

$$3x + y - z = 3$$

Solve the system of equations  $2x - 8y + z = -5$  using Gauss elimination method.

$$x - 2y + 9z = 8$$

**Solution**

The given system is equivalent to

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$$

$$A \quad X = B$$

the augmented matrix is

$$[A|B] = \left[ \begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array} \right]$$

now making A as an upper triangular matrix

$$\sim \left[ \begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & -7 \\ 0 & \frac{-7}{3} & \frac{28}{3} & / \end{array} \right] \quad R_2 - \frac{2}{3}R_1, R_3 - \frac{1}{3}R_1$$

now choosing  $\frac{-26}{3}$  as the pivot from the second column,

$$\sim \left[ \begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & -7 \\ 0 & 0 & \frac{693}{78} & \frac{231}{26} \end{array} \right] \quad R_2 - \frac{2}{3}R_1, R_3 - \frac{1}{3}R_1$$

from this we get

$$3x + y - z = 3$$

$$\frac{-26}{3}y + \frac{5}{3}z = -7$$

$$\frac{693}{78}z = \frac{231}{26}$$

now by back substitution  $z = 1$

$$\frac{-26}{3}y = -7 - \frac{5}{3}(z) = -7 - \frac{5}{3}(1) = \frac{-26}{3} \Rightarrow y = 1$$

$$\text{now } x = \frac{1}{3}[3 - y + z] = \frac{1}{3}[3 - 1 + 1] = 1$$

so the solution is  $x = y = z = 1$

### Example

Using Gauss elimination method, solve the system.

$$3.15x - 1.96y + 3.85z = 12.95$$

$$2.13x + 5.12y - 2.89z = -8.61$$

$$5.92x + 3.05y + 2.15z = 6.88$$

### Solution

The given system is equivalent to

$$\begin{bmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

$$A \quad X = B$$

$$[A|B] = \left[ \begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 6.7335 & -5.0855 & -17.4578 \end{array} \right] \quad R_2 - \frac{2.13}{3.15}R_1, R_3 - \frac{5.92}{3.15}R_1$$

choosing 6.4453 as pivot

$$\sim \left[ \begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 0 & 0.6534 & 0.6853 \end{array} \right] \quad R_3 - \frac{6.7335}{6.4453}R_2$$

from this, we get

$$3.15x - 1.96y + 3.85z = 12.95$$

$$6.4453y - 5.4933z = -17.3667$$

$$0.6534z = 0.6853$$

by backward substitution

$$z = \frac{0.6853}{0.6534} = 1.0488215$$

$$y = \frac{5.4933 - 17.3667}{6.4453} = -1.8005692$$

$$x = \frac{1.96y - 3.85z + 12.95}{3.15} = 1.708864$$

### Example

Solve the system of equation

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_1 + x_2 + 3x_3 - 2x_4 = -6$$

$$2x_1 + 3x_2 - x_3 + 2x_4 = 7$$

$$x_1 + 2x_2 + x_3 - x_4 = -2$$

By using Gauss elimination method.

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_1 + x_2 + 3x_3 - 2x_4 = -6$$

$$2x_1 + 3x_2 - x_3 + 2x_4 = 7$$

$$x_1 + 2x_2 + x_3 - x_4 = -2$$

the given system in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \\ -2 \end{bmatrix}$$

$$A \quad X = B$$

$$[A|B] = \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{array} \right]$$

$$\sim \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & 0 & -2 & -4 \end{array} \right] \begin{matrix} \\ R_2 - R_1, R_3 - 2R_1, R_4 - R_1 \end{matrix}$$

since the element in the second column is zero so interchange the rows

$$\sim \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & 0 & -2 & -4 \end{array} \right] \begin{matrix} \\ R_{23} \end{matrix}$$

$$\sim \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & 3 & -2 & -4 \end{array} \right] \begin{matrix} \\ R_4 - R_2 \end{matrix}$$

now the pivot is 2, therefore

$$\sim \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & 0 & 5 & 5 \end{array} \right] \begin{matrix} \\ R - \frac{3}{2}R \\ 4 \quad 2 \quad 3 \end{matrix}$$

from this we get

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_2 - 3x_3 = 3$$

$$2x_3 - 3x_4 = -8$$

$$\left(\frac{5}{2}\right)x_4 = 5$$

$$\text{now } x_4 = 2$$

$$x_3 = \frac{1}{2}(-8 + 3x_4) = \frac{1}{2}(-8 + 6) = -1$$

now

$$x_2 = 3 + 3x_3 = 3 - 3 = 0$$

now from equation 1

$$x_1 = 2 - x_2 - x_3 - x_4 = 2 - 0 + 1 - 2 = 1$$

$$\text{so } x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 2$$

**Solution of Linear System of Equations and Matrix Inversion Gauss–****Jordan Elimination Method**

This method is a variation of Gaussian elimination method. In this method, the elements above and below the diagonal are simultaneously made zero. That is a given system is reduced to an equivalent diagonal form using elementary transformations. Then the solution of the resulting diagonal system is obtained. Sometimes, we normalize the pivot row with respect to the pivot element, before elimination. Partial pivoting is also used whenever the pivot element becomes zero.

**Example**

Solve the system of equations using Gauss-Jordan elimination method:

$$\begin{cases} x + 2y + z = 8 \\ 2x + 3y + 4z = 20 \\ 4x + 3y + 2z = 16 \end{cases}$$

**Solution**

In matrix notation, the given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 16 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -16 \end{pmatrix} \quad (-2) R1 + R2 \text{ and } (-4) R1 + R3$$

Now, we eliminate y from the first and third rows using the second row. Thus, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ -36 \end{pmatrix}$$

Before, eliminating z from the first and second row, normalize the third row with respect to the pivot element, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ 3 \end{pmatrix}$$

Using the third row of above Equation, and eliminating z from the first and second rows, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

The solution is

$$x = 1, y = 2, z = 3.$$

**Example**

Solve the system of equation by using the Gauss Jordan elimination method

$$10x + y + z = 12$$

$$2x + 10y + z = 13$$

$$x + y + 5z = 7$$

**Solution**

$$10x + y + z = 12$$

$$2x + 10y + z = 13$$

$$x + y + 5z = 7$$

the given system

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

the augmented matrix may be written as

$$\begin{aligned} [A/B] &= \left[ \begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array} \right] \\ &\sim \left[ \begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array} \right] \begin{array}{l} R - 9R \\ \\ \end{array} \\ &\sim \left[ \begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{array} \right] \begin{array}{l} \\ R - 2R, R - R \\ \\ \end{array} \\ &\sim \left[ \begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 1 & 89 & 59 \\ 0 & 9 & 49 & 58 \end{array} \right] \begin{array}{l} \\ R - 3R \\ \\ \end{array} \\ &\sim \left[ \begin{array}{ccc|c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & -473 & -473 \end{array} \right] \begin{array}{l} \\ R + 8R, R - 9R \\ \\ \end{array} \\ &\sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{array}{l} \\ \\ \\ \end{array} \end{aligned}$$

thus the system reduces to reduced echlon form

$$\text{so } x = y = z = 1$$

### Example

Solve the system of equations by Gauss Jordan method

$$10x_1 + x_2 + x_3 = 12$$

$$x_1 + 10x_2 - x_3 = 10$$

$$x_1 - 2x_2 + 10x_3 = 9$$

### Solution

$$10x_1 + x_2 + x_3 = 12$$

$$x_1 + 10x_2 - x_3 = 10$$

$$x_1 - 2x_2 + 10x_3 = 9$$

the matrix form of the given system may be written as

$$\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$$

the augmented matrix may be written as

$$\begin{aligned} [A/B] &= \left[ \begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{array} \right] \\ &\sim \left[ \begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 1 & 10 & -1 & 88 \\ 1 & -2 & 10 & 87 \end{array} \right] \begin{matrix} \\ R - 9R \\ \end{matrix} \\ &\sim \left[ \begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 & 8 \\ 0 & 1 & 0 & 1 \end{array} \right] \begin{matrix} \\ R - R, R - R \\ \end{matrix} \\ &\sim \left[ \begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 & 8 \\ 0 & 1 & 0 & 1 \end{array} \right] \begin{matrix} \\ R_2, R_3 \\ \end{matrix} \\ &\sim \left[ \begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right] \begin{matrix} \\ R - 8R \\ \end{matrix} \\ &\sim \left[ \begin{array}{ccc|c} 1 & 0 & -79 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{matrix} \\ R - 89R, R - R \\ \end{matrix} \\ &\sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \begin{matrix} \\ R + 79R, R + R \\ \end{matrix} \\ &\sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \end{aligned}$$

so the system gives the values of all the three variables

$$x_1 = x_2 = x_3 = 1$$

### Example

Solve the system of equations by using Gauss Jordan method.

$$x + 2y + z - w = -2$$

$$2x + 3y - z + 2w = 7$$

$$x + y + 3z - 2w = -6$$

$$x + y + z + w = 2$$

### Solution

$$x + 2y + z - w = -2$$

$$2x + 3y - z + 2w = 7$$

$$x + y + 3z - 2w = -6$$

$$x + y + z + w = 2$$

the system may be written as

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -6 \\ 2 \end{bmatrix}$$

the augmented matrix may be written as

$$\begin{aligned} [A/B] &= \left[ \begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \end{array} \right] \\ &\sim \left[ \begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 0 & 1 & 3 & -4 & 11 \\ 0 & -1 & 2 & -1 & -4 \\ 0 & -1 & 0 & 2 & 4 \end{array} \right] \begin{array}{l} R_2 - 2R_1, R_3 - R_1, R_4 - R_1 \end{array} \\ &\sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 7 & 20 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 3 & -2 & -7 \end{array} \right] \begin{array}{l} R_1 - 2R_2, \frac{1}{5}(R_3 + R_2), R_4 + R_2 \end{array} \\ &\sim \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 + 5R_3, R_2 - 3R_3, R_4 - 3R_3 \end{array} \\ &\sim \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 2R_4, R_2 + R_4, R_3 + R_4 \end{array} \end{aligned}$$

the system maybe written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

the values of all the ariables are

$$x = 1, y = 0, z = -1, w = 2$$

### Crout's ReductionMethod

Here the coefficient matrix [A] of the system of equations is decomposed into the product of two matrices [L] and [U], where [L] is a lower-triangular matrix and [U] is an upper-triangular matrix with 1's on its main diagonal.

For the purpose of illustration, consider a general matrix in the form

$$[L][U] = [A]$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The sequence of steps for getting [L] and [U] are given below:

Step I: Multiplying all the rows of [L] by the first column of [U], we get

$$l_{11} = a_{11}, \quad l_{21} = a_{21}, \quad l_{31} = a_{31}$$

Thus, we observe that the first column of [L] is same as the first column of [A].

Step II: Multiplying the first row of [L] by the second and third columns of [U], we obtain

$$l_{11}u_{12} = a_{12}, \quad l_{11}u_{13} = a_{13}$$

Or

$$u_{12} = \frac{a_{12}}{l_{11}}, \quad u_{13} = \frac{a_{13}}{l_{11}}$$

Thus, the first row of [U] is obtained. We continue this process, getting alternately the column of [L] and a row of [U].

Step III: Multiply the second and third rows of [L] by the second column of [U] to get

$$l_{21}u_{12} + l_{22} = a_{22}, \quad l_{31}u_{12} + l_{32} = a_{32}$$

This gives

$$l_{22} = a_{22} - l_{21}u_{12}, \quad l_{32} = a_{32} - l_{31}u_{12}$$

Step IV: Now, multiply the second row of [L] by the third column of [U] which yields

$$l_{21}u_{13} + l_{22}u_{23} = a_{23}$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

Step V: Lastly, we multiply the third row of [L] by the third column of [U] and get

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}$$

This gives

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Thus, the above five steps determine [L] and [U].

This algorithm can be generalized to any linear system of order n.

Consider a system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

In matrix notation as [A](X) = (B). Let [A] = [L] [U], then we get,

$$[L][U](X) = (B)$$

Substituting [U] (X) = (Z) in Eq. we obtain [L] (Z) = (B)

$$\begin{cases} l_{11}z_1 = b_1 \\ l_{21}z_1 + l_{22}z_2 = b_2 \\ l_{31}z_1 + l_{32}z_2 + l_{33}z_3 = b_3 \end{cases}$$

Having computed z1, z2 and z3, we can compute x1, x2, and x3 from equation [U] (X) = (Z) or from

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

This method is also known as Cholesky reduction method. This technique is widely used in the numerical solutions of partial differential equations.

This method is very popular from computer programming point of view, since the storages space reserved for matrix [A] can be used to store the elements of [L] and [U] at the end of computation.

This method fails if any

$$a_{ii} = 0$$

### Example

Solve the following system of equations by Crout's reduction method

$$5x_1 - 2x_2 + x_3 = 4$$

$$7x_1 + x_2 - 5x_3 = 8$$

$$3x_1 + 7x_2 + 4x_3 = 10$$

### Solution

Let the coefficient matrix [A] be written as [L] [U]. Thus,

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Step I: Multiply all the rows of [L] by the first column of [U], we get

$$l_{11} = 5, \quad l_{21} = 7, \quad l_{31} = 3$$

Step II: Multiply the first row of [L] by the second and third columns of [U], we have

$$l_{11}u_{12} = -2, \quad l_{11}u_{13} = 1$$

$$u_{12} = -\frac{2}{5}, \quad u_{13} = \frac{1}{5}$$

STEP III: Multiply the 2nd and 3rd rows of [L] by the 2nd column of [U], we get

$$\begin{aligned} l_{21}u_{12} + l_{22} &= 1 \quad \text{or} \quad l_{22} = 1 + \frac{14}{5} = \frac{19}{5} \\ l_{31}u_{12} + l_{32} &= 7 \quad \text{or} \quad l_{32} = 7 + \frac{14}{5} = \frac{49}{5} \end{aligned}$$

STEP IV: Multiply the 2nd row of [L] by the 3rd column of [U]

$$l_{21}u_{13} + l_{22}u_{23} = -5$$

$$\frac{19}{5}u_{23} = -5 - \frac{7}{5}$$

$$u_{23} = -\frac{32}{19}$$

STEP V: Finally, multiply the 3rd row of [L] with the 3rd column of [U], we obtain

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4$$

$$l_{33} = \frac{327}{19}$$

Thus, the given system of equations takes the form [L][U][X] = (B).

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{49}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{32}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

That is,

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{5}{19} & 0 \\ 3 & \frac{5}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Let  $[U](X) = (Z)$ , then

$$[L](Z) = (4 \ 8 \ 10)^T$$

Or

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{5}{19} & 0 \\ 3 & \frac{5}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Which gives utilizing these values of  $z$ , the Eq becomes

By back substitution method, we obtain

$$x_3 = \frac{46}{327}, \quad x_2 = \frac{284}{327}, \quad x_1 = \frac{366}{327}$$

This is the required solution.

### Example

$$\begin{aligned} 2x - 3y + 10z &= 3 \\ -x + 4y + 2z &= 20 \\ 5x + 2y + z &= -12 \end{aligned}$$

### Solution

$$2x - 3y + 10z = 3$$

$$-x + 4y + 2z = 20$$

$$5x + 2y + z = -12$$

the given system is  $AX = B$

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

let  $LU = A$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

here

$$u_{11} = 2, u_{12} = -3, u_{13} = 10$$

$$l_{21} u_{11} = -1 \Rightarrow l_{21} = \frac{-1}{2}$$

$$l_{31} u_{11} = 5 \Rightarrow l_{31} = \frac{5}{2}$$

$$l_{21} u_{12} + u_{22} = 4 \Rightarrow u_{22} = 4 - l_{21} u_{12} = 4 - \left(\frac{-1}{2}\right)(-3) = \frac{5}{2}$$

$$l_{21} u_{13} + u_{23} = 2 \Rightarrow u_{23} = 2 - l_{21} u_{13} = 2 - \left(\frac{-1}{2}\right)(10) = 7$$

$$l_{31} u_{12} + l_{32} u_{22} = 5 \Rightarrow l_{32} = \frac{1}{u_{22}} [5 - l_{31} u_{12}] = \frac{19}{5}$$

$$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 0 \Rightarrow u_{33} = 1 - l_{31} u_{13} - l_{32} u_{23} = \frac{253}{5}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & \frac{-253}{5} \end{bmatrix}$$

let  $UX = Y$  where  $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ , then  $LY = B$ ,

$$\text{i.e.} \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

now eqn (1) implies

$$y_1 = 3$$

$$\frac{-1}{2} y_1 + y_2 = 20 \Rightarrow y_2 = \frac{43}{2}$$

$$\frac{5}{2} y_1 + \frac{19}{5} y_2 + y_3 = -12 \Rightarrow y_3 = \frac{-506}{5}$$

$$2x - 3y + 10z = 3, \frac{5}{2}y + 7z = \frac{43}{2}, -\frac{253}{5}z = \frac{-506}{3}$$

by back substitution  $x = -4, y = 3$  and  $z = 2$

### Example

Solve the following system

$$x + 3y + 4z = 4$$

$$x + 4y + 3z = -2$$

$$x + 3y + 4z = 1$$

### Solution

$$x + 3y + 4z = 4$$

$$x + 4y + 3z = -2$$

$$x + 3y + 4z = 1$$

the given system is  $AX = B$

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

let  $LU = A$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

here

$$u_{11} = 1, u_{12} = 3, u_{13} = 4$$

$$l_{21}u_{11} = -1 \Rightarrow l_{21} = -1$$

$$l_{31}u_{11} = 1 \Rightarrow l_{31} = 1$$

$$l_{21}u_{12} + u_{22} = 4 \Rightarrow u_{22} = 4 - l_{21}u_{12} = 4 - \left(-\frac{1}{2}\right)(-3) = \frac{5}{2}$$

$$l_{21}u_{13} + u_{23} = 2 \Rightarrow u_{23} = 4 - l_{21}u_{13} = 4 - (1)3 = 1$$

$$l_{31}u_{12} + l_{32}u_{22} + u_{33} = 3 \Rightarrow l_{32} = \frac{1}{u_{22}}[3 - l_{31}u_{12}] = 0$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} \Rightarrow u_{33} = 4 - l_{31}u_{13} - l_{32}u_{23} = -4$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

let  $UX = Y$  where  $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ , then  $LY = B$ ,

$$\text{i.e.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & & \\ \frac{-1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

now eqn (1) implies

$$y_1 = 4$$

$$y_2 = -2$$

$$y_1 + y_3 = 1 \Rightarrow y_3 = 1 - y_1 = -3$$

we also have

$$x + 3y + 8z = 4$$

$$y - 5z = -2$$

$$-4z = -3$$

by back substituton

$$z = \frac{3}{4}$$

$$y = -2 + 5z = -2 + 5\left(\frac{3}{4}\right) = \frac{7}{4}$$

$$x = 4 - 3y - 8z = 4 - 3\left(\frac{7}{4}\right) - 8\left(\frac{3}{4}\right) = \frac{29}{4}$$

## Solution of Linear System of Equations and Matrix Inversion

### Jacobi's Method

This is an iterative method, where initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way.

In general, when the coefficient matrix of the system of equations is a sparse matrix (many elements are zero), iterative methods have definite advantage over direct methods in respect of economy of computer memory

Such sparse matrices arise in computing the numerical solution of partial differential equations

Let us consider

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array} \right\}$$

In this method, we assume that the coefficient matrix  $[A]$  is strictly diagonally dominant, that is, in each row of  $[A]$  the modulus of the diagonal element exceeds the sum of the off-diagonal elements.

We also assume that the diagonal element do not vanish. If any diagonal element vanishes, the equations can always be rearranged to satisfy this condition.

Now the above system of equations can be written as

$$\left. \begin{array}{l} x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \dots - \frac{a_{1n}}{a_{11}}x_n \\ x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \dots - \frac{a_{2n}}{a_{22}}x_n \\ \vdots \\ x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \dots - \frac{a_{n(n-1)}}{a_{nn}}x_{n-1} \end{array} \right\}$$

We shall take this solution vector  $(x_1, x_2, \dots, x_n)^T$  as a first approximation to the exact solution of system. For convenience, let us denote the first approximation vector by  $(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$  got after taking as an initial starting vector.

Substituting this first approximation in the right-hand side of system, we obtain the second approximation to the given system in the form

$$\left. \begin{aligned} x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \dots - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\ x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \dots - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\ &\vdots \\ x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \dots - \frac{a_{nn-1}}{a_{nn}} x_{n-1}^{(1)} \end{aligned} \right\}$$

This second approximation is substituted into the right-hand side of Equations and obtain the third approximation and so on.

This process is repeated and  $(r+1)th$  approximation is calculated

$$\left. \begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(r)} - \dots - \frac{a_{1n}}{a_{11}} x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(r)} - \dots - \frac{a_{2n}}{a_{22}} x_n^{(r)} \\ &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(r)} - \dots - \frac{a_{nn-1}}{a_{nn}} x_{n-1}^{(r)} \end{aligned} \right\}$$

Briefly, we can rewrite these Equations as

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{a_{ij}}{a_{ii}} x_j^{(r)},$$

$$r = 1, 2, \dots, \quad i = 1, 2, \dots, n$$

It is also known as method of simultaneous displacements,

since no element of  $x_i^{(r+1)}$  is used in this iteration until every element is computed.

A sufficient condition for convergence of the iterative solution to the exact solution is

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|, \quad i = 1, 2, \dots, n$$

When this condition (diagonal dominance) is true, Jacobi's

method converges

### Example

Find the solution to the following system of equations using Jacobi's iterative method for the first five iterations:

$$83x + 11y - 4z = 95$$

$$7x + 52y + 13z = 104$$

$$3x + 8y + 29z = 71$$

### Solution

$$\begin{cases} x = \frac{95}{104} - \frac{11}{83}y + \frac{4}{83}z \\ y = \frac{52}{71} - \frac{52}{3}x - \frac{52}{8}z \\ z = \frac{29}{29} - \frac{29}{29}x - \frac{29}{29}y \end{cases}$$

Taking the initial starting of solution vector as  $(0,0,0)^T$ , from Eq. ,we have the first approximation as

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{pmatrix} = \begin{pmatrix} 1.1446 \\ 2.0000 \\ 2.4483 \end{pmatrix}$$

Now, using Eq. ,the second approximation is computed from the equations

$$\begin{cases} x^{(2)} = 1.1446 - 0.1325y^{(1)} + 0.0482z^{(1)} \\ y^{(2)} = 2.0 - 0.1346x^{(1)} - 0.25z^{(1)} \\ z^{(2)} = 2.4483 - 0.1035x^{(1)} - 0.2759y^{(1)} \end{cases}$$

Making use of the last two equations we get the second approximation as

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{pmatrix} = \begin{pmatrix} 0.9976 \\ 1.2339 \\ 1.7424 \end{pmatrix}$$

Similar procedure yields the third, fourth and fifth approximations to the required solution and they are tabulated as below;

| Variables            |        |        |        |
|----------------------|--------|--------|--------|
| Iteration number $r$ | $x$    | $y$    | $z$    |
| 1                    | 1.1446 | 2.0000 | 2.4483 |
| 2                    | 0.9976 | 1.2339 | 1.7424 |
| 3                    | 1.0651 | 1.4301 | 2.0046 |

|   |        |        |        |
|---|--------|--------|--------|
| 4 | 1.0517 | 1.3555 | 1.9435 |
| 5 | 1.0587 | 1.3726 | 1.9655 |

**Example**

Solve the system by jacobi's iterative method

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

(Perform only four iterations)

**Solution**

Consider the given system as

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

*the system is diagonally dominant*

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

*we start with an initial approximation  $x_0 = y_0 = z_0 = 0$*

*substituting these*

*first iteration*

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y_1 = \frac{1}{11}[33 - 4(0) + 0] = 3$$

$$z_1 = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.916667$$

*Second iteration*

$$x_2 = \frac{1}{8}[20 + 3(3) - 2(2.916667)] = 2.895833$$

$$y_2 = \frac{1}{11}[33 - 4(2.5) + 2.916667] = 2.3560606$$

$$z_2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = 0.9166666$$

*third iteration*

$$x_3 = \frac{1}{8} [20 + 3(2.3560606) - 2(0.9166666)] = 3.1543561$$

$$y_3 = \frac{1}{11} [33 - 4(2.8958333) + 0.9166666] = 2.030303$$

$$z_3 = \frac{1}{12} [35 - 6(2.8958333) - 3(2.3560606)] = 0.8797348$$

*fourth iteration*

$$x_4 = \frac{1}{8} [20 + 3(2.030303) - 2(0.8797348)] = 3.0419299$$

$$y_4 = \frac{1}{11} [33 - 4(3.1543561) + 0.8797348] = 1.9329373$$

$$z_4 = \frac{1}{12} [35 - 6(3.1543561) - 3(2.030303)] = 0.8319128$$

### Example

Solve the system by jacobi's iterative method

$$3x + 4y + 15z = 54.8$$

$$x + 12y + 3z = 39.66$$

$$10x + y - 2z = 7.74$$

(Perform only four iterations)

### Solution

Consider the given system as

$$3x + 4y + 15z = 54.8$$

$$x + 12y + 3z = 39.66$$

$$10x + y - 2z = 7.74$$

*the system is not diagonally dominant we rearrange the system*

$$10x + y - 2z = 7.74$$

$$x + 12y + 3z = 39.66$$

$$3x + 4y + 15z = 54.8$$

$$x = \frac{1}{10} [7.74 - y + 2z]$$

$$y = \frac{1}{12} [39.66 - x - 3z]$$

$$z = \frac{1}{15} [54.8 - 3x - 4y]$$

we start with an initial approximation  $x_0 = y_0 = z_0 = 0$

substituting these

first iteration

$$x_1 = \frac{1}{10} [7.74 - (0) + 2(0)] = 0.774$$

$$y_1 = \frac{1}{12} [39.66 - (0) - 3(0)] = 1.1383333$$

$$z_1 = \frac{1}{15} [54.8 - 3(0) - 4(0)] = 3.6533333$$

Second iteration

$$x_2 = \frac{1}{10} [7.74 - 1.1383333 + 2(3.6533333)] = 1.3908333$$

$$y_2 = \frac{1}{12} [39.66 - 0.774 - 3(3.6533333)] = 2.3271667$$

$$z_2 = \frac{1}{15} [54.8 - 3(0.774) - 4(1.1383333)] = 3.1949778$$

third iteration

$$x_3 = \frac{1}{10} [7.74 - 2.3271667 + 2(3.1949778)] = 1.1802789$$

$$y_3 = \frac{1}{12} [39.66 - 1.3908333 - 3(3.1949778)] = 2.3903528$$

$$z_3 = \frac{1}{15} [54.8 - 3(1.3908333) - 4(2.3271667)] = 2.7545889$$

fourth iteration

$$x_4 = \frac{1}{10} [7.74 - 2.5179962 + 2(2.7798501)] = 1.0781704$$

$$y_4 = \frac{1}{12} [39.66 - 1.1802789 - 3(2.7545889)] = 2.51779962$$

$$z_4 = \frac{1}{15} [54.8 - 3(1.1802789) - 4(2.3903528)] = 2.7798501$$

### Solution of Linear System of Equations and Matrix Inversion Gauss–Seidel Iteration Method

It is another well-known iterative method for solving a system of linear equations of the form

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \right\}$$

In Jacobi's method, the  $(r + 1)$ th approximation to the above system is given by Equations

$$\left. \begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}}x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(r)} - \cdots - \frac{a_{2n}}{a_{22}}x_n^{(r)} \\ \vdots &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(r)} - \cdots - \frac{a_{nn-1}}{a_{nn}}x_{n-1}^{(r)} \end{aligned} \right\}$$

Here we can observe that no element of  $x_i^{(r+1)}$  replaces  $x_i^{(r)}$  entirely for the next cycle of computation.

In Gauss-Seidel method, the corresponding elements of  $x_i^{(r+1)}$  replaces those of  $x_i^{(r)}$  as soon as they become available.

Hence, it is called the method of successive displacements. For illustration consider

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \right\}$$

In Gauss-Seidel iteration, the  $(r + 1)$ th approximation or iteration is computed from:

$$\left. \begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}}x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(r+1)} - \cdots - \frac{a_{2n}}{a_{22}}x_n^{(r)} \\ \vdots &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(r+1)} - \cdots - \frac{a_{nn-1}}{a_{nn}}x_{n-1}^{(r+1)} \end{aligned} \right\}$$

Thus, the general procedure can be written in the following compact form

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^i \frac{a_{ij}}{a_{ii}} x_j^{(r+1)} - \sum_{j=i+1}^n \frac{a_{ij}}{a_{ii}} x_j^{(r)} \quad \text{for all } i=1, 2, \dots, n \text{ and } r=1, 2, \dots$$

To describe system in the first equation, we substitute the r-th approximation into the right-hand side and denote the result by  $x_1^{(r+1)}$ . In the second equation, we substitute

$(x_1^{(r+1)}, x_2^{(r)}, \dots, x_n^{(r)})$  and denote the result by  $x_2^{(r+1)}$

In the third equation, we substitute  $(x_1^{(r+1)}, x_2^{(r+1)}, x_3^{(r)}, \dots, x_n^{(r)})$  and denote the result by

$x_3^{(r+1)}$ , and so on. This process is continued till we arrive at the desired result. For illustration, we consider the following example :

### Note

**The difference between jacobi's method and gauss Seidel method is that in jacobi's method the approximation calculated are used in the next iteration for next approximation but in Gauss-seidel method the new approximation calculated is instantly replaced by the previous one.**

### Example

Find the solution of the following system of equations using Gauss-Seidel method and perform the first five iterations:

$$\begin{aligned} 4x_1 - x_2 - x_3 &= 2 \\ -x_1 + 4x_2 - x_4 &= 2 \\ -x_1 + 4x_3 - x_4 &= 1 \\ -x_2 - x_3 + 4x_4 &= 1 \end{aligned}$$

### Solution

The given system of equations can be rewritten as

$$\left. \begin{aligned} x_1 &= 0.5 + 0.25x_2 + 0.25x_3 \\ x_2 &= 0.5 + 0.25x_1 + 0.25x_4 \\ x_3 &= 0.25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 0.25 + 0.25x_2 + 0.25x_3 \end{aligned} \right\}$$

Taking  $x_2 = x_3 = x_4 = 0$  on the right-hand side of the first equation of the system, we get  $x_1^{(1)} = 0.5$ . Taking  $x_3 = x_4 = 0$  and the current value of  $x_1$ , we get from the 2nd equation of the system

$$x_2^{(1)} = 0.5 + (0.25)(0.5) + 0 = 0.625$$

Further, we take  $x_4 = 0$  and the current value of  $x_1$  we obtain from the third equation of the system

$$\begin{aligned} x_3^{(1)} &= 0.25 + (0.25)(0.5) + 0 \\ &= 0.375 \end{aligned}$$

Now, using the current values of  $x_2$  and  $x_3$  the fourth equation of system gives

$$x_4^{(1)} = 0.25 + (0.25)(0.625) \\ + (0.25)(0.375) = 0.5$$

The Gauss-Seidel iterations for the given set of equations can be written as

$$x_1^{(r+1)} = 0.5 + 0.25x_2^{(r)} + 0.25x_3^{(r)} \\ x_2^{(r+1)} = 0.5 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)} \\ x_3^{(r+1)} = 0.25 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)} \\ x_4^{(r+1)} = 0.25 + 0.25x_2^{(r+1)} + 0.25x_3^{(r+1)}$$

Now, by Gauss-Seidel procedure, the 2nd and subsequent approximations can be obtained and the sequence of the first five approximations are tabulated as below:

| Iteration<br>number r | Variables |         |         |         |
|-----------------------|-----------|---------|---------|---------|
|                       | x1        | x2      | x3      | x4      |
| 1                     | 0.5       | 0.625   | 0.375   | 0.5     |
| 2                     | 0.75      | 0.8125  | 0.5625  | 0.59375 |
| 3                     | 0.84375   | 0.85938 | 0.60938 | 0.61719 |
| 4                     | 0.86719   | 0.87110 | 0.62110 | 0.62305 |
| 5                     | 0.87305   | 0.87402 | 0.62402 | 0.62451 |

### Example

Solve the system by Gauss-Seidel iterative method

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

(Perform only four iterations)

### Solution

Consider the given system as

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

*the system is diagonally dominant*

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

*we start with an initial approximation  $x_0 = y_0 = z_0 = 0$*

*substituting these*

*first iteration*

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y_1 = \frac{1}{11}[33 - 4(2.5) + 0] = 2.0909091$$

$$z_1 = \frac{1}{12}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

*Second iteration*

$$x_2 = \frac{1}{8}[20 + 3y_1 - z_1] = \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1] = \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2] = \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = 0.9141701$$

*third iteration*

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

$$y_3 = \frac{1}{11}[33 - 4(3.0266228) + 0.9141701] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825163)] = 0.9077262$$

*fourth iteration*

$$x_4 = \frac{1}{8} [20 + 3(1.9825163) - 2(0.9077262)] = 3.0165121$$

$$y_4 = \frac{1}{11} [33 - 4(3.0165121) + 0.9077262] = 1.9856071$$

$$z_4 = \frac{1}{12} [35 - 6(3.0165121) - 3(1.9856071)] = 0.8319128$$

### Example

Solve the system by using Gauss-seidel iteration method

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

### Solution

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

*the given system is diagonally dominant so we will make it diagonally dominant by interchanging the equations*

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

*hence we can apply Gauss – Seidel method*

*from the above equations*

$$x = \frac{1}{28} [32 - 4y + z]$$

$$y = \frac{1}{17} [35 - 2x - 4z]$$

$$z = \frac{1}{10} [24 - x - 3y]$$

*First approximation*

putting  $y = z = 0$

$$x_1 = \frac{1}{28}[32] = 1.1428571$$

putting  $x = 1.1428571$  ,  $z = 0$

$$y_1 = \frac{1}{17}[35 - 2(1.1428571) - 4(0)] = 1.9243697$$

putting  $x = 1.1428571$  ,  $y = 1.9243697$

$$z_1 = \frac{1}{10}[24 - 1.1428571 - 3(1.9243697)] = 1.7084034$$

*Second iteration*

$$x_2 = \frac{1}{28}[32 - 4(1.9243697) + 1.7084034] = 0.9289615$$

$$y_2 = \frac{1}{17}[35 - 2(0.9289615) - 4(1.7084034)] = 1.5475567$$

$$z_2 = \frac{1}{10}[24 - 0.9289615 - 3(1.5475567)] = 1.8408368$$

*third iteration*

$$x_3 = \frac{1}{28}[32 - 4(1.5475567) + 1.8428368] = 0.9875932$$

$$y_3 = \frac{1}{17}[35 - 2(0.9875932) - 4(1.8428368)] = 1.5090274$$

$$z_3 = \frac{1}{10}[24 - 0.9875932 - 3(1.5090274)] = 1.8485325$$

*fourth iteration*

$$x_4 = \frac{1}{28}[32 - 4(1.5090274) + 1.8485325] = 0.9933008$$

$$y_4 = \frac{1}{17}[35 - 2(0.9933008) - 4(1.8428368)] = 1.5070158$$

$$z_4 = \frac{1}{10}[24 - 0.9933008 - 3(1.5070158)] = 1.8485652$$

### Example

Using Gauss-Seidel iteration method, solve the system of the equation.

$$10x - 2y - z - w = 3$$

$$-2x + 10y - z - w = 15$$

$$-x - y + 10z - 2w = 27$$

$$-x - y - 2z + 10w = -9$$

(Perform only four iterations)

**Solution**

$$10x - 2y - z - w = 3$$

$$-2x + 10y - z - w = 15$$

$$-x - y + 10z - 2w = 27$$

$$-x - y - 2z + 10w = -9$$

it is diagonally dominant and we may write equation as

$$x = \frac{1}{10}[3 + 2y + z + w]$$

$$y = \frac{1}{10}[15 + 2x + z + w]$$

$$z = \frac{1}{10}[27 + x + y + 2w]$$

$$w = \frac{1}{10}[-9 + x + y + 2z]$$

first approximation

putting  $y = z = w = 0$  on RHS of (1), we get

$$x_1 = 0.3$$

$$y_1 = \frac{1}{10}[15 + 2(0.3)] = 1.56$$

putting  $x = 0.3, y = 1.56$  and  $w = 0$

$$z_1 = \frac{1}{10}[27 + 0.3 + 1.56] = 2.886$$

putting  $x = 0.3, y = 1.56$  and  $z = 2.886$

$$w_1 = \frac{1}{10}[-9 + 0.3 + 1.56 + 2(2.886)] = -0.1368$$

second iteration

$$x_2 = \frac{1}{10}[3 + 2(1.56) + 2.886 - 0.1368] = 0.88692$$

$$y_2 = \frac{1}{10}[15 + 2(0.88692) + 2.886 - 0.1368] = 1.952304$$

$$z_2 = \frac{1}{10}[27 + 0.88692 + 1.952304 + 2(-0.1368)] = 2.9565624$$

$$w_2 = \frac{1}{10}[-9 + 0.88692 + 1.952304 + 2(2.9565624)] = -0.0247651$$

third iteration

$$x_3 = \frac{1}{10}[3 + 2(1.952304) + 2.9565624 - 0.0247651] = 0.9836405$$

$$y_3 = \frac{1}{10} [15 + 2(0.9836405) + 2.9565624 - 0.0247651] = 1.9899087$$

$$z_3 = \frac{1}{10} [27 + 0.9836405 + 1.9899087 + 2(-0.0247651)] = 2.9924019$$

$$w_3 = \frac{1}{10} [-9 + 0.983405 + 1.9899087 + 2(2.9924019)] = -0.0041647$$

*fourth iteration*

$$x_4 = \frac{1}{10} [3 + 2(1.9899087) + 2.9924019 - 0.0041647] = 0.9968054$$

$$y_4 = \frac{1}{10} [15 + 2(0.9968054) + 2.9924019 - 0.0041647] = 1.9981848$$

$$z_4 = \frac{1}{10} [27 + 0.9968054 + 1.9981848 + 2(-0.0041647)] = 2.9986661$$

$$w_4 = \frac{1}{10} [-9 + 0.9968054 + 1.9981848 + 2(2.9986661)] = -0.0007677$$

#### **Note**

When to stop the iterative processes, we stop the iterative process when we get the required accuracy means if you are asked that find the accurate up to four places of decimal then we will simply perform up to that iteration after which we will get the required accuracy. If we calculate the root of the equation and its consecutive values are 1.895326125, 1.916366125, 1.919356325, 1.919326355, 1.919327145, 1.919327128. Here the accuracy up to seven places of decimal is achieved so if you are asked to acquire the accuracy up to six places of decimal then we will stop here. But in the solved examples only some iteration are carried out and accuracy is not considered here.

## Solution of Linear System of Equations and Matrix Inversion

### Relaxation Method

This is also an iterative method and is due to Southwell. To explain the details, consider again the system of equations

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array} \right\}$$

Let

$$X^{(p)} = (x_1^{(p)}, x_2^{(p)}, \dots, x_n^{(p)})^T$$

be the solution vector obtained iteratively after p-th iteration. If  $R_i^{(p)}$  denotes the residual of the i-th equation of system given above, that is of  $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$  defined by

$$R_i^{(p)} = b_i - a_{i1}x_1^{(p)} - a_{i2}x_2^{(p)} - \cdots - a_{in}x_n^{(p)}$$

we can improve the solution vector successively by reducing the largest residual to zero at that iteration. This is the basic idea of relaxation method.

To achieve the fast convergence of the procedure, we take all terms to one side and then reorder the equations so that the largest negative coefficients in the equations appear on the diagonal.

Now, if at any iteration,  $R_i$  is the largest residual in magnitude, then we give an increment to  $x_i$ ;  $a_{ii}$  being the coefficient of  $x_i$

$$dx_i = \frac{R_i}{a_{ii}}$$

In other words, we change  $x_i$  to  $(x_i + dx_i)$  to relax  $R_i$  that is to reduce  $R_i$  to zero.

### Example

Solve the system of equations

$$6x_1 - 3x_2 + x_3 = 11$$

$$2x_1 + x_2 - 8x_3 = -15$$

$$x_1 - 7x_2 + x_3 = 10$$

by the relaxation method, starting with the vector (0, 0, 0).

### Solution

At first, we transfer all the terms to the right-hand side and reorder the equations, so that the largest coefficients in the equations appear on the diagonal.

Thus, we get

$$\left. \begin{aligned} 0 &= 11 - 6x_1 + 3x_2 - x_3 \\ 0 &= 10 - x_1 + 7x_2 - x_3 \\ 0 &= -15 - 2x_1 - x_2 + 8x_3 \end{aligned} \right\}$$

after interchanging the 2nd and 3rd equations.

Starting with the initial solution vector  $(0, 0, 0)$ , that is taking  $x_1 = x_2 = x_3 = 0$ ,

we find the residuals  $R_1 = 11$ ,  $R_2 = 10$ ,  $R_3 = -15$

of which the largest residual in magnitude is  $R_3$ , i.e. the 3rd equation has more error and needs immediate attention for improvement.

Thus, we introduce a change,  $dx_3$  in  $x_3$  which is obtained from the formula

$$dx_3 = -\frac{R_3}{a_{33}} = \frac{15}{8} = 1.875$$

Similarly, we find the new residuals of large magnitude and relax it to zero, and so on. We shall continue this process, until all the residuals are zero or very small.

| Iteration | Residuals | Maximum | Difference | Variables |
|-----------|-----------|---------|------------|-----------|
|-----------|-----------|---------|------------|-----------|

| number | R1     | R2     | R3      | $R_i$  | $dx_i$  | x1     | x2 | x3    |
|--------|--------|--------|---------|--------|---------|--------|----|-------|
| 0      | 11     | 10     | -15     | -15    | 1.875   | 0      | 0  | 0     |
| 1      | 9.125  | 8.125  | 0       | 9.125  | 1.5288  | 0      | 0  | 1.875 |
| 2      | 0.0478 | 6.5962 | -3.0576 | 6.5962 | -0.9423 | 1.5288 | 0  | 1.875 |

| Iteration | Residuals | Maximum | Difference | Variables |
|-----------|-----------|---------|------------|-----------|
|-----------|-----------|---------|------------|-----------|

| number | R1      | R2     | R3      | $\ddot{U}$ | $\ddot{U}$                        | x1     | x2      | x3    |
|--------|---------|--------|---------|------------|-----------------------------------|--------|---------|-------|
| 0      | 11      | 10     | -15     | -15        | $\frac{15}{8}$<br>=1.875          | 0      | 0       | 0     |
| 1      | 9.125   | 8.125  | 0       | 9.125      | $\frac{-9.125}{(-6)}$<br>=1.5288  | 0      | 0       | 1.875 |
| 2      | 0.0478  | 6.5962 | -3.0576 | 6.5962     | $\frac{-6.5962}{7}$<br>=-0.9423   | 1.5288 | 0       | 1.875 |
| 3      | -2.8747 | 0.0001 | -2.1153 | -2.8747    | $\frac{2.8747}{(-6)}$<br>=-0.4791 | 1.0497 | -0.9423 | 1.875 |
| 4      | -0.0031 | 0.4792 | -1.1571 | -1.1571    | $\frac{1.1571}{8}$<br>=0.1446     | 1.0497 | -0.9423 | 1.875 |

| Iteration |         | Residuals |        | Maximum | Difference | Variables              |        |         |        |
|-----------|---------|-----------|--------|---------|------------|------------------------|--------|---------|--------|
| number    | R1      | R2        | R3     |         | $\ddot{U}$ | $\ddot{U}$             | x1     | x2      | x3     |
| 5         | -0.1447 | 0.3346    | 0.0003 | 0.3346  |            | -.3346/7<br>=-0.0478   | 1.0497 | -0.9423 | 2.0196 |
| 6         | 0.2881  | 0.0000    | 0.0475 | 0.2881  |            | -.2881/(-6)<br>=0.0480 | 1.0497 | -0.9901 | 2.0196 |
| 7         | -0.0001 | 0.048     | 0.1435 | 0.1435  |            | =-0.0179               | 1.0017 | -0.9901 | 2.0196 |
| 8         | 0.0178  | 0.0659    | 0.0003 | -       |            | -                      | 1.0017 | -0.9901 | 2.0017 |

At this stage, we observe that all the residuals  $R1$ ,  $R2$  and  $R3$  are small enough and therefore we may take the corresponding values of  $x_i$  at this iteration as the solution. Hence, the numerical solution is given by

$$x_1 = 1.0017, \quad x_2 = -0.9901, \quad x_3 = 2.0017,$$

The exact solution is

$$x_1 = 1.0, \quad x_2 = -1.0, \quad x_3 = 2.0$$

### Example

Solve by relaxation method, the equation

$$10x - 2y - 2z = 6$$

$$-x - 10y - 2z = 7$$

$$-x - y + 10z = 8$$

### Solution

The residual  $r_1, r_2, r_3$  are given by

$$r_1 = 6 - 10x + 2y + 2z$$

$$r_2 = 7 + x - 10y + 2z$$

$$r_3 = 8 + x + y - 10z$$

The operation table is as follows

| x | y | z | r1  | r2  | r3  |    |
|---|---|---|-----|-----|-----|----|
| 1 | 0 | 0 | -10 | 1   | 1   | L1 |
| 0 | 1 | 0 | 2   | -10 | 1   | L2 |
| 0 | 0 | 1 | 2   | 2   | -10 | L3 |

The relaxation table is as follows

| x | y | z | r1 | r2 | r3 |          |
|---|---|---|----|----|----|----------|
| 0 | 0 | 0 | 6  | 7  | 8  | L4       |
| 0 | 0 | 1 | 8  | 9  | -2 | L5=L4+L3 |
| 0 | 1 | 0 | 10 | -1 | -1 | L6=L5+L2 |
| 1 | 0 | 0 | 0  | 0  | 0  | L7=L6+L1 |

### Explanation

(1) In L4, the largest residual is 8. to reduce it, To reduce it, we give an increment of

$$\frac{8}{c_3} = \frac{8}{10} = 0.8 \cong 1$$

the resulting residuals are obtained by

$$L_4 + (1)L_3, \text{ i.e line } L_5$$

(2) In line  $L_5$  the largest residual is 9

$$\text{Increment} = \frac{9}{b_2} = \frac{9}{10} = 0.9 \cong 1$$

The resulting residuals ( $= L_6$ )  $= L_5 + 1.L_2$

(3) In line  $L_6$ , the largest residual is 10

$$\text{Increment} = \frac{10}{a_1} = \frac{10}{10} \cong 1$$

The resulting residuals ( $= L_6$ )  $= L_5 + 1.L_2$

Exact solution is arrived and it is  $x=1, y=1, z=1$

### Example

Solve the system by relaxation method, the equations

$$9x - y + 2z = 7$$

$$x + 10y - 2z = 15$$

$$2x - 2y - 13z = -17$$

### Solution

The residuals  $r_1, r_2, r_3$  are given by

$$9x - y + 2z = 9$$

$$x + 10y - 2z = 15$$

$$2x - 2y - 13z = -17$$

here

$$r_1 = 9 - 9x + y - 2z$$

$$r_2 = 15 - x - 10y + 2z$$

$$r_3 = -17 - 2x + 2y + 13z$$

Operation table

| x | y | z | r1 | r2  | r3 |
|---|---|---|----|-----|----|
| 1 | 0 | 0 | -9 | -1  | -2 |
| 0 | 1 | 0 | 1  | -10 | 2  |
| 0 | 0 | 1 | -2 | 2   | 13 |

Relaxation table is

| x    | y     | z    | r1    | r2   | r3     |
|------|-------|------|-------|------|--------|
| 0    | 0     | 0    | 9     | 15   | -17    |
| 0    | 0     | 1    | 7     | 17   | -4     |
| 0    | 1     | 0    | 8     | 7    | -2     |
| 0.89 | 0     | 0    | -0.01 | 6.11 | -3.78  |
| 0    | 0.61  | 0    | 0.6   | 0.01 | -2.56  |
| 0    | 0     | 0.19 | 0.22  | 0.39 | -0.09  |
| 0    | 0.039 | 0    | 0.259 | 0    | -0.012 |

|       |   |         |          |          |          |
|-------|---|---------|----------|----------|----------|
| 0.028 | 0 | 0       | 0.007    | -0.028   | -0.068   |
| 0     | 0 | 0.00523 | -0.00346 | -1.01754 | -0.00001 |

Then  $x=0.89+0.028=0.918$ ;  $y=1+0.61+0.039=1.694$

And  $z=1+0.19+0.00523=1.19523$

Now substituting the values of  $x, y, z$  in (1), we get

$r_1=9-9(0.918)+1.649-2(1.19523)=-0.00346$

$r_2=15-0.918-10(1.649)+2(1.19523)=-0.1754$

$r_3=-17-2(0.918)+2(1.649)+13(1.19523)=-0.00001$

Which is agreement with the final residuals.

## Solution of Linear System of Equations and Matrix Inversion

### Matrix Inversion

Consider a system of equations in the form

$$[A](X) = (B)$$

One way of writing its solution is in the form

$$(X) = [A]^{-1}(B)$$

Thus, the solution to the system can also be obtained if the inverse of the coefficient matrix  $[A]$  is known. That is the product of two square matrices is an identity matrix

$$[A][B] = [I]$$

$$\text{then, } [B] = [A]^{-1}$$

$$\text{and } [A] = [B]^{-1}$$

Every square non-singular matrix will have an inverse.

Gauss elimination and Gauss-Jordan methods are popular among many methods available for finding the inverse of a matrix.

### Gaussian Elimination Method

In this method, if  $A$  is a given matrix, for which we have to find the inverse; at first, we place an identity matrix, whose order is same as that of  $A$ , adjacent to  $A$  which we call an augmented matrix.

Then the inverse of  $A$  is computed in two stages. In the first stage,  $A$  is converted into an upper triangular form, using Gaussian elimination method

In the second stage, the above upper triangular matrix is reduced to an identity matrix by row transformations. All these operations are also performed on the adjacently placed identity matrix.

Finally, when  $A$  is transformed into an identity matrix, the adjacent matrix gives the inverse of  $A$ .

In order to increase the accuracy of the result, it is essential to employ partial pivoting.

### Example

Use the Gaussian elimination method to find the inverse of the matrix

$$A = \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ \hline & & & 3 & 5 & 3 \end{array} \right]$$

### Solution

At first, we place an identity matrix of the same order adjacent to the given matrix. Thus, the augmented matrix can be written as

$$\left[ \begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & \\ 4 & 3 & -1 & 0 & 1 & 0 & \\ \hline 3 & 5 & 3 & 0 & 0 & 1 & \end{array} \right]$$

**Stage I** (Reduction to upper triangular form): Let  $R_1$ ,  $R_2$  and  $R_3$  denote the 1st, 2nd and 3rd rows of a matrix. In the 1st column, 4 is the largest element, thus interchanging  $R_1$  and  $R_2$  to bring the pivot element 4 to the place of  $a_{11}$ , we have the augmented matrix in the form

$$\left[ \begin{array}{cccccc|c} 4 & 3 & -1 & 0 & 1 & 0 & \\ 1 & 1 & 1 & 1 & 0 & 0 & \\ \hline 3 & 5 & 3 & 0 & 0 & 1 & \end{array} \right]$$

Divide  $R_1$  by 4 to get

$$\left[ \begin{array}{cccccc|c} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & \\ 3 & 5 & 3 & 0 & 0 & 1 & \end{array} \right]$$

Perform  $R_2 - R_1 \rightarrow$ , which gives

$$\left[ \begin{array}{cccccc|c} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 & \\ \hline 3 & 5 & 3 & 0 & 0 & 1 & \\ \hline \end{array} \right]$$

Perform  $R_3 - 3R_1 \rightarrow R_3$  in the above equation, which yields

$$\left[ \begin{array}{cccccc|c} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 & \\ \hline 0 & \frac{5}{4} & \frac{11}{4} & 0 & -\frac{3}{4} & 1 & \\ \hline \end{array} \right]$$

Now, looking at the second column for the pivot, the max  $(1/4, 11/4)$  is  $11/4$ . Therefore, we interchange  $R_2$  and  $R_3$  in the last equation and get

$$\left[ \begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{3}{4} & 0 & -\frac{1}{4} & 1 \\ 0 & \frac{1}{4} & \frac{1}{5} & 1 & -\frac{1}{4} & 0 \end{array} \right]$$

Now, divide  $R_2$  by the pivot  $a_{22} = 11/4$ , and obtain

$$\left[ \begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{3}{11} & 0 & -\frac{1}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{1}{5} & 1 & -\frac{1}{4} & 0 \end{array} \right]$$

Performing  $R_3 - (1/4)R_2 \rightarrow R_3$  yields

$$\left[ \begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{3}{11} & 0 & -\frac{1}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{1}{11} & -\frac{1}{11} \end{array} \right]$$

Finally, we divide  $R_3$  by  $(10/11)$ , thus getting an upper triangular form

$$\left[ \begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{3}{11} & 0 & -\frac{1}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

## Stage II

Reduction to an identity matrix  $(1/4)R_3 + R_1$  and  $(-15/11)R_3 + R_2$

$$\left[ \begin{array}{cccccc} 1 & \frac{3}{4} & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

Finally, performing  $R_1 - (3/4)R_2 \rightarrow R_1$  we obtain

$$\begin{bmatrix} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Thus, we have

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

### Example

Find the inverse of the  $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$  using gauss elimination method.

### Solution

We have the augmented system

$$[A/I] = \left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 - (-)R_1, R_3 - (-)R_1 \\ R_2 - \frac{3}{2}R_1, R_3 - \frac{1}{2}R_1 \end{array}$$

$$\sim \left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & \frac{5}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{7}{2} & \frac{7}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & \frac{5}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array} \right] \begin{array}{l} R_3 - 7R_2 \end{array}$$

$$\text{now if } \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

then the system 1 is the inverse of the given matrix the system is equivalent to three systems

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ \frac{2}{10} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix}$$

$$\begin{array}{lll} x_{11} = -3 & x_{21} = 12 & x_{31} = -5 \\ x_{21} = \frac{5}{2} & x_{22} = \frac{-17}{2} & x_{23} = \frac{7}{2} \\ x_{31} = \frac{-1}{2} & x_{22} = \frac{3}{2} & x_{23} = -\frac{1}{2} \end{array}$$

and the inverse matrix is

$$\begin{array}{l} 1 \left[ \begin{array}{ccc|c} -6 & 5 & -1 & 1 \\ 24 & -17 & 3 & 0 \end{array} \right] \\ 2 \left[ \begin{array}{ccc|c} -10 & 7 & -1 & 0 \end{array} \right] \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

by back substitution, the three systems of equation may be written as

### Example

Fine the inverse of the matrix  $\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$  using gauss elimination method.

### Solution

$$[A/I] = \left[ \begin{array}{cccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{cccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 & 0 \\ 0 & -9 & 3 & -1 & 0 & 1 & 0 \end{array} \right] \begin{array}{l} R_2 - \frac{1}{2}R_1, R_3 - \frac{1}{4}R_1 \\ R_2 - \frac{1}{2}R_1, R_3 - \frac{1}{4}R_1 \end{array}$$

$$\left[ \begin{array}{cccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 & 0 \\ 0 & -9 & 3 & -1 & 0 & 1 & 0 \end{array} \right]$$

now if  $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$  is the inverse of the given matrix, then the system (1) is

equivalent to three systems

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{lll} x_{11} = \frac{-4}{3} & x_{21} = \frac{5}{3} & x_{31} = \frac{7}{3} \\ x_{21} = \frac{2}{7} & x_{22} = \frac{-2}{-8} & x_{23} = \frac{-3}{10} \\ x_{31} = \frac{7}{3} & x_{22} = \frac{-8}{3} & x_{23} = \frac{-10}{3} \end{array}$$

and the inverse matrix is

$$\left[ \begin{array}{ccc|ccc} 1 & -\frac{4}{3} & 2 & \frac{7}{3} & 0 & 0 \\ 0 & 5 & -8 & -1 & 1 & 0 \\ 0 & 3 & -2 & 3 & 0 & 1 \\ 0 & 7 & -3 & -10 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 \end{array} \right] = \frac{1}{3} \left[ \begin{array}{ccc|ccc} -4 & 6 & 7 & 0 & 0 & 0 \\ 5 & -6 & -8 & 0 & 0 & 0 \\ 7 & -9 & -10 & 0 & 0 & 0 \end{array} \right]$$

### Gauss - Jordan Method

This method is similar to Gaussian elimination method, with the essential difference that the stage I of reducing the given matrix to an upper triangular form is not needed.

However, the given matrix can be directly reduced to an identity matrix using elementary row operations.

### Example

Find the inverse of the given matrix by Gauss-Jordan method

$$A = \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 3 & 0 & 1 \end{array} \right]$$

### Solution

Let  $R_1$ ,  $R_2$  and  $R_3$  denote the 1st, 2nd and 3rd rows of a matrix. We place the identity matrix adjacent to the given matrix. So the augmented matrix is given by

$$\left[ \begin{array}{cccccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 5 & 3 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Performing  $R_2 - 4R_1 \rightarrow R_2$ , we get

$$\left[ \begin{array}{cccccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -5 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 5 & 3 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Now, performing  $R_3 - 3R_1 \rightarrow R_3$ , we obtain

$$\left[ \begin{array}{cccccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 & 0 & 0 & 1 & 1 \end{array} \right]$$

Carrying out further operations  $R_2 + R_1 \rightarrow R_1$  and  $R_3 + 2R_2 \rightarrow R_3$ ,  
we arrive at

$$\left[ \begin{array}{cccccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array} \right]$$

Now, dividing the third row by  $-10$ , we get

$$\left[ \begin{array}{cccccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

Further, we perform  $R_1 + 4R_3 \rightarrow R_1$ , and  $R_2 + 5R_3 \rightarrow R_2$  to get

$$\left[ \begin{array}{cccccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & -1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

Finally, multiplying  $R_2$  by  $-1$ , we obtain

$$\left[ \begin{array}{cccccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

Hence, we have

$$A^{-1} = \left[ \begin{array}{ccc} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

### Exercise

Solve all the above examples solved by Gauss elimination by using gauss Jordan method.

### Eigen Value Problems

Let  $[A]$  be an  $n \times n$  square matrix. Suppose, there exists a scalar  $\lambda$  and a vector

$$X = (x_1 \ x_2 \ \dots \ x_n)_T$$

such that

$$[A](X) = \lambda(X)$$

$$\frac{d}{dx}(e^{ax}) = a(e^{ax})$$

$$\frac{d^2}{dx^2}(\sin ax) = -a^2(\sin ax)$$

Then  $\lambda$  is the eigen value and  $X$  is the corresponding eigenvector of the matrix  $[A]$ .

We can also write it as  $[A - \lambda I](X) = (O)$

This represents a set of  $n$  homogeneous equations possessing non-trivial solution, provided

$$|A - \lambda I| = 0$$

This determinant, on expansion, gives an  $n$ -th degree polynomial which is called characteristic polynomial of  $[A]$ , which has  $n$  roots. Corresponding to each root, we can solve these equations in principle, and determine a vector called eigenvector.

Finding the roots of the characteristic equation is laborious. Hence, we look for better methods suitable from the point of view of computation. Depending upon the type of matrix  $[A]$  and on what one is looking for, various numerical methods are available.

### Power Method and Jacobi's Method

#### Note!

We shall consider only real and real-symmetric matrices and discuss power and Jacobi's methods

#### Power Method

To compute the largest eigen value and the corresponding eigenvector of the system

$$[A](X) = \lambda(X)$$

where  $[A]$  is a real, symmetric or un-symmetric matrix, the power method is widely used in practice.

#### Procedure

**Step 1:** Choose the initial vector such that the largest element is unity.

**Step 2:** The normalized vector  $v^{(0)}$  is pre-multiplied by the matrix  $[A]$ .

**Step 3:** The resultant vector is again normalized.

**Step 4:** This process of iteration is continued and the new normalized vector is repeatedly pre-multiplied by the matrix  $[A]$  until the required accuracy is obtained.

At this point, the result looks like

$$u^{(k)} = [A]v^{(k-1)} = q_k v^{(k)}$$

Here,  $q_k$  is the desired largest eigen value and  $v^{(k)}$  is the corresponding eigenvector.

### Example

Find the eigen value of largest modulus, and the associated eigenvector of the matrix by power method

$$[A] = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$$

### Solution

We choose an initial vector  $v^{(0)}$

as  $(1,1,1)^T$ .

Then, compute first iteration

$$u^{(1)} = [A]v^{(0)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 14 \end{pmatrix}$$

Now we normalize the resultant vector to get

$$u^{(1)} = 14 \begin{pmatrix} \frac{1}{2} \\ \frac{6}{7} \\ 1 \end{pmatrix} = q_1 v^{(1)}$$

The second iteration gives,

$$\begin{aligned} u^{(2)} &= [A]v^{(1)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{6}{7} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{39}{7} \\ \frac{67}{7} \\ \frac{171}{14} \end{pmatrix} \\ &= 12.2143 \begin{pmatrix} 0.456140 \\ 0.783626 \\ 1.0 \end{pmatrix} = q_2 v^{(2)} \end{aligned}$$

Continuing this procedure, the third and subsequent iterations are given in the following slides

$$u^{(3)} = [A]v^{(2)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} 0.456140 \\ 0.783626 \\ 1.0 \end{pmatrix}$$

$$\begin{aligned}
 &= \begin{pmatrix} 5.263158 \\ 9.175438 \\ 11.935672 \end{pmatrix} = 11.935672 \begin{pmatrix} 0.44096 \\ 0.776874 \\ 1.0 \end{pmatrix} = q_3 v^{(3)} \\
 u^{(4)} &= [A]v^{(3)} = \begin{pmatrix} 5.18814 \\ 9.07006 \\ 11.86036 \end{pmatrix} \\
 &= 11.8636 \begin{pmatrix} 0.437435 \\ 0.764737 \\ 1.0 \end{pmatrix} = q_4 v^{(4)} \\
 u^{(5)} &= [A]v^{(4)} = \begin{pmatrix} 5.16908 \\ 9.04395 \\ 11.84178 \end{pmatrix} \\
 &= 11.84178 \begin{pmatrix} 0.436512 \\ 0.763732 \\ 1.0 \end{pmatrix} = q_5 v^{(5)}
 \end{aligned}$$

After rounding-off, the largest eigen value and the corresponding eigenvector as accurate to two decimals are

$$\lambda = 11.84 \quad (X) = \begin{pmatrix} 0.44 \\ 0.76 \\ 1.00 \end{pmatrix}$$

### Example

Find the first three iterations of the power method of the given matrix

$$\begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix}$$

we choose finitial vector as  $v^{(0)} = (1,1,1)^t$

first iteration

$$u^{(1)} = [A]v^{(0)} = \begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7+6-3 \\ -12-20+24 \\ -6-12+16 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \\ -2 \end{bmatrix}$$

$$\text{by diagonalising } g \begin{bmatrix} 10 \\ -8 \\ -2 \end{bmatrix} \quad u^{(1)} = 10 \begin{bmatrix} 1 \\ -0.8 \\ -0.2 \end{bmatrix} = q_1 v^{(1)}$$

second iteration

$$u^{(2)} = [A]v^{(1)} = \begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 7-4.8+0.6 \\ -12+16-4.8 \\ -6+9.6-3.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -0.8 \\ 0.4 \end{bmatrix}$$

$$\text{by diagonalising } g \begin{bmatrix} 2.8 \\ -0.8 \\ 0.4 \end{bmatrix} \quad u^{(1)} = 2.8 \begin{bmatrix} 1 \\ -0.2857 \\ 0.1428 \end{bmatrix} = q_2 v^{(2)}$$

third iteration

$$u^{(3)} = [A]v^{(2)} = \begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -0.2857 \\ 0.1428 \end{bmatrix} = \begin{bmatrix} 7-1.7142-0.4284 \\ -12+5.714+3.4272 \\ -6+3.4284+2.2848 \end{bmatrix} = \begin{bmatrix} 4.8574 \\ -2.8588 \\ -0.2868 \end{bmatrix}$$

now daigonalising

$$\begin{bmatrix} 4.8574 \\ -2.8588 \\ -0.2868 \end{bmatrix} \quad \text{now normalising } 4.8574 \begin{bmatrix} 1 \\ -0.5885 \\ -0.0590 \end{bmatrix}$$

### Example

Find the first three iteration of the power method applied on the following matrices

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{use } x^0 = (-1, 2, 1)^t$$

**Solution**

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \text{ USE } x^{(0)} = (-1, 2, 1)^t$$

1st iterations

$$u^{(1)} = [A]x^{(0)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-2+0 \\ 2+8-2 \\ 0-2+2 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

now we normalize the resultant vector to get

$$u^{(1)} = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = 8 \begin{bmatrix} -3/8 \\ 1 \\ 0 \end{bmatrix} = q_1 x^{(1)}$$

$$u^{(2)} = [A]x^{(1)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3/8 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/8 -1 + 0 \\ -6/8 + 4 + 0 \\ 0 - 1 + 0 \end{bmatrix} = \begin{bmatrix} -1.375 \\ 4.75 \\ -1 \end{bmatrix}$$

$$u^{(2)} = \begin{bmatrix} -1.375 \\ 4.75 \\ -1 \end{bmatrix} = 4.75 \begin{bmatrix} -0.28947 \\ 1 \\ -0.2152 \end{bmatrix}$$

$$u^{(3)} = [A]x^{(2)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.28947 \\ 1 \\ -0.2152 \end{bmatrix} = \begin{bmatrix} -1.28947 \\ 4.99998 \\ -1.42104 \end{bmatrix} = 4.99998 \begin{bmatrix} -0.25789 \\ 1 \\ -0.28420 \end{bmatrix}$$

**Exercise**

Find the largest eigen value and the corresponding eigen vector by power method after fourth iteration starting with the initial vector  $v^{(0)} = (0, 0, 1)^T$

$$[A] = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$

Let

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

be the distinct eigen values of an  $n \times n$  matrix  $[A]$ , such that  $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$  and

suppose  $v_1, v_2, \dots, v_n$  are the corresponding eigen vectors

Power method is applicable if the above eigen values are real and distinct, and hence, the corresponding eigenvectors are linearly independent.

Then, any eigenvector  $v$  in the space spanned by the eigenvectors  $v_1, v_2, \dots, v_n$

can be written as their linear combination  $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

Pre-multiplying by  $A$  and substituting

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2, \quad \dots \quad Av_n = \lambda_n v_n$$

We get

$$Av = \lambda_1 \left( c_1 v_1 + c_2 \frac{\lambda_2}{\lambda_1} v_2 + \dots + c_n \frac{\lambda_n}{\lambda_1} v_n \right)$$

Again, pre-multiplying by  $A$  and simplifying, we obtain

$$A^2 v = \lambda_1^2 \left[ c_1 v_1 + c_2 \left( \frac{\lambda_2}{\lambda_1} \right)^2 v_2 + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^2 v_n \right]$$

Similarly, we have

$$A^r v = \lambda_1^r \left[ c_1 v_1 + c_2 \left( \frac{\lambda_2}{\lambda_1} \right)^r v_2 + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^r v_n \right]$$

and

$$A^{r+1} v = (\lambda_1)^{r+1} \left[ c_1 v_1 + c_2 \left( \frac{\lambda_2}{\lambda_1} \right)^{r+1} v_2 + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^{r+1} v_n \right]$$

Now, the eigen value  $\lambda_1$

can be computed as the limit of the ratio of the corresponding components of  $A^r v$  and  $A^{r+1} v$ .

That is,

$$\lambda_1 = \frac{\lambda_1^{r+1}}{\lambda_1^r} = \lim_{r \rightarrow \infty} \frac{(A^{r+1}v)_p}{(A^r v)_p}, \quad p = 1, 2, \dots, n$$

Here, the index  $p$  stands for the  $p$ -th component in the corresponding vector. Sometimes, we may be interested in finding the least eigen value and the corresponding eigenvector.

In that case, we proceed as follows.

We note that  $[A](X) = \lambda(X)$ .

Pre-multiplying by  $[A^{-1}]$ , we get

$$[A^{-1}][A](X) = [A^{-1}]\lambda(X) = \lambda[A^{-1}](X)$$

Which can be rewritten as

$$[A^{-1}](X) = \frac{1}{\lambda}(X)$$

which shows that the inverse matrix has a set of eigen values which are the reciprocals of the eigen values of  $[A]$ .

Thus, for finding the eigen value of the least magnitude of the matrix  $[A]$ , we have to apply power method to the inverse of  $[A]$ .

## Jacobi's Method

### Definition

An  $n \times n$  matrix  $[A]$  is said to be orthogonal if

$$[A]^T [A] = [I],$$

$$\text{i.e. } [A]^T = [A]^{-1}$$

In order to compute all the eigen values and the corresponding eigenvectors of a real symmetric matrix, Jacobi's method is highly recommended. It is based on an important property from matrix theory, which states

that, if  $[A]$  is an  $n \times n$  real symmetric matrix, its eigen values are real, and there exists an orthogonal matrix  $[S]$  such that the diagonal matrix  $D$  is

$$[S^{-1}][A][S]$$

This digitalization can be carried out by applying a series of orthogonal transformations

$$S_1, S_2, \dots, S_n,$$

Let  $A$  be an  $n \times n$  real symmetric matrix. Suppose  $|a_{ij}|$  be numerically the largest element amongst the off-diagonal elements of  $A$ . We construct an orthogonal matrix  $S_1$  defined as

$$s_{ij} = -\sin\theta, \quad s_{ji} = \sin\theta,$$

$$s_{ii} = \cos\theta, \quad s_{jj} = \cos\theta$$

While each of the remaining off-diagonal elements are zero, the remaining diagonal elements are assumed to be unity. Thus, we construct  $S_1$  as under

$$S_1 = \begin{matrix} & \begin{matrix} \text{i-th column} & \text{j-th column} \end{matrix} \\ & \begin{matrix} \downarrow & \downarrow \end{matrix} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \cos\theta & \cdots & -\sin\theta & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \sin\theta & \cdots & \cos\theta & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} & \begin{matrix} \leftarrow \text{i-th row} \\ \\ \leftarrow \text{j-th row} \end{matrix} \end{matrix}$$

Where  $\cos\theta, -\sin\theta, \sin\theta$  and  $\cos\theta$  are inserted in  $(i, i), (i, j), (j, i), (j, j)$ -th positions respectively, and elsewhere it is identical with a unit matrix.

Now, we compute

$$D = S_1^{-1} A S_1 = S_1^T A S_1$$

Since  $S_1$  is an orthogonal matrix, such that  $S_1^{-1} = S_1^T$ . After the transformation, the elements at the position  $(i, j), (j, i)$  get annihilated, that is  $d_{ij}$  and  $d_{ji}$  reduce to zero, which is seen as follows:

$$\begin{aligned}
 & \begin{bmatrix} d_{ii} & d_{ij} \\ d_{ji} & d_{jj} \end{bmatrix} \\
 &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
 &= \begin{bmatrix} a_{ii} \cos^2\theta + a_{jj} \sin^2\theta & (a_{ii} - a_{jj}) \sin\theta \cos\theta \\ (a_{jj} - a_{ii}) \sin\theta \cos\theta & a_{ii} \sin^2\theta + a_{jj} \cos^2\theta \end{bmatrix}
 \end{aligned}$$

Therefore,  $d_{ij} = 0$  only if,

$$a_{ij} \cos 2\theta + \frac{a_{jj} - a_{ii}}{2} \sin 2\theta = 0$$

That is if

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}}$$

Thus, we choose  $\theta$  such that the above equation is satisfied, thereby, the pair of off-diagonal elements  $d_{ij}$  and  $d_{ji}$  reduces to zero. However, though it creates a new pair of zeros, it also introduces non-zero contributions at formerly zero positions.

Also, the above equation gives four values of  $\theta$ , but to get the least possible rotation, we choose

**Example**

Find all the eigen values and the corresponding eigen vectors of the matrix by Jacobi's method

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

**Solution**

The given matrix is real and symmetric. The largest off-diagonal element is found to be  $a_{13} = a_{31} = 2$ .

Now, we compute

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{4}{0} = \infty$$

This gives,  $\theta = \pi/4$

Thus, we construct an orthogonal matrix  $S_1$  as

$$S_1 = \begin{bmatrix} \cos \frac{\pi}{4} & 0 & -\sin \frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin \frac{\pi}{4} & 0 & \cos \frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The first rotation gives,

$$\begin{aligned} D &= S_1^{-1} A S_1 \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

We observe that the elements  $d_{13}$  and  $d_{31}$  got annihilated. To make sure that calculations are correct up to this step, we see that the sum of the diagonal elements of  $D_1$  is same as the sum of the diagonal elements of the original matrix  $A$ .

As a second step, we choose the largest off-diagonal element of  $D_1$  and is found to be

$d_{12} = d_{21} = 2$ , and compute

$$\tan 2\theta = \frac{2d_{12}}{d_{11} - d_{22}} = \frac{4}{0} = \infty$$

This again gives  $\theta = \pi/4$

Thus, we construct the second rotation matrix as

$$S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

At the end of the second rotation, we get

$$\begin{aligned} D &= S_2^{-1} D S_2 \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

This turned out to be a diagonal matrix, so we stop the computation. From here, we notice that the eigen values of the given matrix are 5, 1 and -1. The eigenvectors are the column vectors of  $S = S_1 S_2$

Therefore

$$\begin{aligned} S &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

### Example

Find all the eigen values of the matrix by Jacobi's method.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

### Solution

Here all the off-diagonal elements are of the same order of magnitude. Therefore, we can choose any one of them. Suppose, we choose  $a_{12}$  as the largest element and compute

$$\tan 2\theta = \frac{-1}{0} = \infty$$

Which gives,  $\theta = \pi/4$ .

Then  $\cos\theta = \sin\theta = 1/\sqrt{2}$

and we construct an orthogonal matrix  $S_1$  such that

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first rotation gives

$$\begin{aligned} D &= S^{-1}AS \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 3 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \end{bmatrix} \end{aligned}$$

Now, we choose  $d_{13} = -1/\sqrt{2}$

As the largest element of D1 and compute

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{-\sqrt{2}}{1 - 2}$$

$$\theta = 27^\circ 22' 41''.$$

Now we construct another orthogonal matrix S2, such that

$$S_2 = \begin{bmatrix} 0.888 & 0 & -0.459 \\ 0 & 1 & 0 \\ 0.459 & 0 & 0.888 \end{bmatrix}$$

At the end of second rotation, we obtain

$$D = S_2^{-1} D S_2 = \begin{bmatrix} 0.634 & -0.325 & 0 \\ 0.325 & 3 & -0.628 \\ 0 & -0.628 & 2.365 \end{bmatrix}$$

Now, the numerically largest off-diagonal element of D2 is found to be  $d_{23} = -0.628$  and compute.

$$\tan 2\theta = \frac{-2 \times 0.628}{3 - 2.365}$$

$$\theta = -31^\circ 35' 24''.$$

Thus, the orthogonal matrix is

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.852 & 0.524 \\ 0 & -0.524 & 0.852 \end{bmatrix}$$

At the end of third rotation, we get

$$D = S^{-1} D S = \begin{bmatrix} 0.634 & -0.277 & 0 \\ 0.277 & 3.386 & 0 \\ 0 & 0 & 1.979 \end{bmatrix}$$

To reduce  $D_3$  to a diagonal form, some more rotations are required. However, we may take 0.634, 3.386 and 1.979 as eigen values of the given matrix.

### Example

Using Jacobi's method, find the eigenvalues and eigenvectors of the following matrix,

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

### Solution:

The given matrix is real and symmetric. The largest off –diagonal element is found to be

$$a_{12} = a_{21} = \frac{1}{2}$$

Now we compute

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2\left(\frac{1}{2}\right)}{1 - \frac{1}{3}} = \frac{3}{2}$$

$$\theta = \frac{\tan^{-1}\left(\frac{3}{2}\right)}{2} = 28.155$$

Thus we construct an orthogonal matrix  $S_1$  as

$$S_1 = \begin{bmatrix} \cos 28.155 & -\sin 28.155 & 0 \\ \sin 28.155 & \cos 28.155 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.882 & -0.472 & 0 \\ 0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first rotation gives,  $D = S_1^{-1} A S_1$

$$\begin{aligned} &= \begin{bmatrix} 0.882 & 0.472 & 0 \\ -0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} 0.882 & -0.472 & 0 \\ 0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.268 & 0.000 & 0.412 \\ 0.000 & 0.066 & 0.063 \\ 0.412 & 0.063 & 0.200 \end{bmatrix} \end{aligned}$$

We see that sum of the diagonal elements of  $D_1 = 1.53$

And the sum of the diagonal elements of  $A = 1.53$

This means that our question is going right.

As a second step we choose the largest of off-diagonal element of  $D_1$ , which is  $d_{13} = d_{31} = 0.412$ , and compute

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{2(0.412)}{1.268 - 0.200} = 0.772$$

$$\theta = \frac{\tan^{-1}(0.772)}{2} = 18.834$$

Thus we construct an orthogonal matrix  $S_2$  as

$$S_2 = \begin{bmatrix} \cos 18.834 & 0 & -\sin 18.834 \\ 0 & 1 & 0 \\ \sin 18.834 & 0 & \cos 18.834 \end{bmatrix} = \begin{bmatrix} 0.946 & 0 & -0.323 \\ 0 & 1 & 0 \\ 0.323 & 0 & 0.946 \end{bmatrix}$$

Thus the rotation gives,

$$D = S_2^{-1} D_1 S_2$$

$$= \begin{bmatrix} 0.946 & 0 & 0.323 \\ 0 & 1 & 0 \\ -0.323 & 0 & 0.946 \end{bmatrix} \begin{bmatrix} 1.268 & 0.000 & 0.412 \\ 0.000 & 0.066 & 0.063 \\ 0.412 & 0.063 & 0.200 \end{bmatrix} \begin{bmatrix} 0.946 & 0 & -0.323 \\ 0 & 1 & 0 \\ 0.323 & 0 & 0.946 \end{bmatrix}$$

$$= \begin{bmatrix} 1.408 & 0.020 & -0.001 \\ 0.020 & 0.066 & 0.060 \\ -0.001 & 0.060 & 0.059 \end{bmatrix}$$

We again see that sum of the diagonal elements of  $D_2 = 1.53$

Also the sum of the diagonal elements of  $A = 1.53$

This means that our question is going right.

Hence the eigenvalues are 1.408, .066 and .059 and the corresponding eigenvectors are the columns of  $S$ . Where

$$S = S_1 S_2$$

$$= \begin{bmatrix} 0.882 & -0.472 & 0 \\ 0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.946 & 0 & -0.323 \\ 0 & 1 & 0 \\ 0.323 & 0 & 0.946 \end{bmatrix}$$

$$= \begin{bmatrix} .8343 & -.472 & -.2848 \\ .446 & .88 & -.1524 \\ .323 & 0 & .946 \end{bmatrix}$$



## Interpolation

### Introduction

Finite differences play an important role in numerical techniques, where tabulated values of the functions are available.

For instance, consider a function  $y = f(x)$ .

As  $x$  takes values  $x_0, x_1, x_2, \dots, x_n$ ,

Let the corresponding values of  $y$  be  $y_0, y_1, y_2, \dots, y_n$ .

That is, for given a table of values,  $(x_k, y_k), k = 0, 1, 2, \dots, n$ ;

The process of estimating the value of  $y$ , for any intermediate value of  $x$ , is called interpolation. The method of computing the value of  $y$ , for a given value of  $x$ , lying outside the table of values of  $x$  is known as extrapolation. If the function  $f(x)$  is known, the value of  $y$  corresponding to any  $x$  can be readily computed to the desired accuracy. For interpolation of a tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

### Finite Difference Operators

#### Forward Differences

For a given table of values  $(x_k, y_k), k = 0, 1, 2, \dots, n$  with equally spaced abscissas of a function  $y = f(x)$ , we define the forward difference operator  $\Delta$  as follows

$$\Delta y_i = y_{i+1} - y_i, \quad i = 0, 1, \dots, (n-1)$$

To be explicit, we write

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ \Delta y_1 &= y_2 - y_1 \\ &\vdots \quad \quad \quad \vdots \\ \Delta y_{n-1} &= y_n - y_{n-1} \end{aligned}$$

These differences are called *first differences of the function  $y$*  and are denoted by the symbol  $\Delta y_i$ . Here,  $\Delta$  is called the first difference operator

Similarly, the differences of the first differences are called second differences, defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

Thus, in general

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

Here  $\Delta^2$  is called the *second difference operator*. Thus, continuing, we can define,  $r$ -th difference of  $y$ , as

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

By defining a difference table as a convenient device for displaying various differences, the above defined differences can be written down systematically by constructing a difference table for values

$$(x_k, y_k), k = 0, 1, \dots, 6$$

#### Forward Difference Table

| $x$   | $y$   | $\Delta y$   | $\Delta^2 y$   | $\Delta^3 y$   | $\Delta^4 y$   | $\Delta^5 y$   | $\Delta^6 y$   |
|-------|-------|--------------|----------------|----------------|----------------|----------------|----------------|
| $x_0$ | $y_0$ | $\Delta y_0$ |                |                |                |                |                |
| $x_1$ | $y_1$ | $\Delta y_1$ | $\Delta^2 y_0$ | $\Delta^3 y_0$ | $\Delta^4 y_0$ |                |                |
| $x_2$ | $y_2$ | $\Delta y_2$ | $\Delta^2 y_1$ | $\Delta^3 y_1$ | $\Delta^4 y_1$ | $\Delta^5 y_0$ |                |
| $x_3$ | $y_3$ | $\Delta y_3$ | $\Delta^2 y_2$ | $\Delta^3 y_2$ | $\Delta^4 y_1$ | $\Delta^5 y_1$ | $\Delta^6 y_0$ |
| $x_4$ | $y_4$ | $\Delta y_4$ | $\Delta^2 y_3$ | $\Delta^3 y_3$ |                |                |                |
| $x_5$ | $y_5$ | $\Delta y_5$ | $\Delta^2 y_4$ |                |                |                |                |
| $x_6$ | $y_6$ |              |                |                |                |                |                |

This difference table is called *forward difference* table or *diagonal difference* table. Here, each difference is located in its appropriate column, midway between the elements of the previous column.

Please note that the subscript remains constant along each diagonal of the table. The first term in the table, that is  $y_0$  is called the leading term, while the differences

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots$  are called leading differences

### Example

Construct a forward difference table for the following values of  $x$  and  $y$ :

| $x$ | 0.1   | 0.3   | 0.5   | 0.7   | 0.9  | 1.1   | 1.3   |
|-----|-------|-------|-------|-------|------|-------|-------|
| $y$ | 0.003 | 0.067 | 0.148 | 0.248 | 0.37 | 0.518 | 0.697 |

**Solution**

| $x$ | $y$   | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|-----|-------|------------|--------------|--------------|--------------|--------------|
| 0.1 | 0.003 |            |              |              |              |              |
| 0.3 | 0.067 | 0.064      | 0.017        |              |              |              |
| 0.5 | 0.148 | 0.081      | 0.019        | 0.002        |              |              |
| 0.7 | 0.248 | 0.1        | 0.022        | 0.003        | 0.001        |              |
| 0.9 | 0.37  | 0.122      | 0.026        | 0.004        | 0.001        | 0            |
| 1.1 | 0.518 | 0.148      | 0.031        | 0.005        | 0.001        | 0            |
| 1.3 | 0.697 | 0.179      |              |              |              |              |

**Example**

Express  $\Delta^2 y_0$  and  $\Delta^3 y_0$  in terms of the values of the function  $y$ .

**Solution:**

Noting that each higher order difference is defined in terms of the lower order difference, we have

$$\begin{aligned}\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) \\ &= y_2 - 2y_1 + y_0\end{aligned}$$

And

$$\begin{aligned}\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0) \\ &= (y_3 - y_2) - (y_2 - y_1) - (y_2 - y_1) + (y_1 - y_0) \\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

Hence, we observe that the coefficients of the values of  $y$ , in the expansion of  $\Delta^2 y_0, \Delta^3 y_0$ , are binomial coefficients.

Thus, in general, we arrive at the following result: -

$$\Delta^n y_0 = y_n - {}^n C_1 y_{n-1} + {}^n C_2 y_{n-2} - {}^n C_3 y_{n-3} + \cdots + (-1)^n y_0$$

**Example**

Show that the value of  $y_n$  can be expressed in terms of the leading value  $y_0$  and the leading differences

$$\Delta y_0, \Delta^2 y_0, \dots, \Delta^n y_0.$$

**Solution**

The forward difference table will be

$$\left. \begin{array}{l} y_1 - y_0 = \Delta y_0 \quad \text{or} \quad y_1 = y_0 + \Delta y_0 \\ y_2 - y_1 = \Delta y_1 \quad \text{or} \quad y_2 = y_1 + \Delta y_1 \\ y_3 - y_2 = \Delta y_2 \quad \text{or} \quad y_3 = y_2 + \Delta y_2 \end{array} \right\}$$

Similarly,

$$\left. \begin{array}{l} \Delta y_2 - \Delta y_1 = \Delta^2 y_1 \quad \text{or} \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \\ \Delta y_1 - \Delta y_0 = \Delta^2 y_0 \quad \text{or} \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0 \end{array} \right\}$$

Similarly, we can also write

$$\left. \begin{array}{l} \Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0 \quad \text{or} \quad \Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0 \\ \Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1} \quad \text{or} \quad \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \end{array} \right\}$$

$$\Delta y_2 = (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$

$$= \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0$$

$$\begin{aligned} y_3 &= y_2 + \Delta y_2 = (y_1 + \Delta y_1) + (\Delta y_1 + \Delta^2 y_1) \\ &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 \\ &= (1 + \Delta)^3 y_0 \end{aligned}$$

Similarly, we can symbolically write

$$y_1 = (1 + \Delta)y_0,$$

$$y_2 = (1 + \Delta)^2 y_0,$$

$$y_3 = (1 + \Delta)^3 y_0$$

.....

$$y_n = (1 + \Delta)^n y_0$$

Hence, we obtain

$$y_n = y_0 + {}^n C_1 \Delta y_0 + {}^n C_2 \Delta^2 y_0 + {}^n C_3 \Delta^3 y_0 + \cdots + \Delta^n y_0$$

OR

$$y_n = \sum_{i=0}^n {}^n C_i \Delta^i y_0$$

## Backward Differences

For a given table of values  $(x_k, y_k)$ ,  $k = 0, 1, 2, \dots, n$  of a function  $y = f(x)$  with equally spaced abscissas, the first backward differences are usually expressed in terms of the backward difference operator  $\nabla$  as

$$\nabla y_i = y_i - y_{i-1}, \quad i = n, (n-1), \dots, 1$$

To be explicit, we write

$$\begin{aligned} \nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \nabla y_n &= y_n - y_{n-1} \end{aligned}$$

The differences of these differences are called *second differences* and they are denoted by  $\nabla^2 y_2, \nabla^2 y_3, \dots, \nabla^2 y_n$ .

$$\begin{aligned} \nabla^2 y_1 &= \nabla y_2 - \nabla y_1 \\ \nabla^2 y_2 &= \nabla y_3 - \nabla y_2 \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \nabla^2 y_n &= \nabla y_n - \nabla y_{n-1} \end{aligned}$$

That is

Thus, in general, the second backward differences are

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}, \quad i = n, (n-1), \dots, 2$$

While the  $k$ -th backward differences are given as

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \quad i = n, (n-1), \dots, k$$

These backward differences can be systematically arranged for a table of values  $(x_k, y_k)$ ,  $k = 0, 1, \dots, 6$  shown below.

**Backward Difference Table**

| $x$   | $y$   | $\nabla y$   | $\nabla^2 y$   | $\nabla^3 y$   | $\nabla^4 y$   | $\nabla^5 y$   | $\nabla^6 y$   |
|-------|-------|--------------|----------------|----------------|----------------|----------------|----------------|
| $x_0$ | $y_0$ |              |                |                |                |                |                |
| $x_1$ | $y_1$ | $\nabla y_1$ | $\nabla^2 y_2$ | $\nabla^3 y_3$ | $\nabla^4 y_4$ |                |                |
| $x_2$ | $y_2$ | $\nabla y_2$ | $\nabla^2 y_3$ | $\nabla^3 y_4$ | $\nabla^4 y_5$ | $\nabla^5 y_5$ |                |
| $x_3$ | $y_3$ | $\nabla y_3$ | $\nabla^2 y_4$ | $\nabla^3 y_5$ | $\nabla^4 y_6$ | $\nabla^5 y_6$ | $\nabla^6 y_6$ |
| $x_4$ | $y_4$ | $\nabla y_4$ | $\nabla^2 y_5$ | $\nabla^3 y_6$ |                |                |                |
| $x_5$ | $y_5$ | $\nabla y_5$ | $\nabla^2 y_6$ |                |                |                |                |
| $x_6$ | $y_6$ | $\nabla y_6$ |                |                |                |                |                |

From this table, it can be observed that the subscript remains constant along every backward diagonal.

### Example

Show that any value of  $y$  can be expressed in terms of  $y_n$  and its backward differences.

**Solution:**

From

$$\nabla y_i = y_i - y_{i-1} \quad i = n, (n-1), \dots, 1$$

We get

$$y_{n-1} = y_n - \nabla y_n \quad y_{n-2} = y_{n-1} - \nabla y_{n-1}$$

From

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}, \quad i = n, (n-1), \dots, 2$$

We get

$$\nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$$

From these equations, we obtain

$$y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$$

Similarly, we can show that

$$y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$$

Symbolically, these results can be rewritten as follows:

$$y_{n-1} = (1 - \nabla) y_n$$

$$y_{n-2} = (1 - \nabla)^2 y_n$$

$$y_{n-3} = (1 - \nabla)^3 y_n$$

.....

$$y_{n-r} = (1 - \nabla)^r y_n$$

$$y_{n-r} = y_n - {}^nC_1 \nabla y_n + {}^nC_2 \nabla^2 y_n - \dots + (-1)^r \nabla^r y_n$$

### Central Differences

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol  $\delta$  to represent central difference operator and the subscript of  $\delta y$  as for any difference as the average of the subscripts

$$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1,$$

In General,

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

These central differences can be systematically arranged as indicated in the Table

| $x$   | $y$   | $\delta y$        | $\delta^2 y$   | $\delta^3 y$       | $\delta^4 y$   | $\delta^5 y$       | $\delta^6 y$   |
|-------|-------|-------------------|----------------|--------------------|----------------|--------------------|----------------|
| $x_0$ | $y_0$ |                   |                |                    |                |                    |                |
| $x_1$ | $y_1$ | $\delta y_{1/2}$  | $\delta^2 y_1$ | $\delta^3 y_{3/2}$ | $\delta^4 y_2$ |                    |                |
| $x_2$ | $y_2$ | $\delta y_{3/2}$  | $\delta^2 y_2$ | $\delta^3 y_{5/2}$ | $\delta^4 y_3$ | $\delta^5 y_{5/2}$ |                |
| $x_3$ | $y_3$ | $\delta y_{5/2}$  | $\delta^2 y_3$ | $\delta^3 y_{7/2}$ | $\delta^4 y_4$ | $\delta^5 y_{7/2}$ | $\delta^6 y_3$ |
| $x_4$ | $y_4$ | $\delta y_{7/2}$  | $\delta^2 y_4$ | $\delta^3 y_{9/2}$ |                |                    |                |
| $x_5$ | $y_5$ | $\delta y_{9/2}$  | $\delta^2 y_5$ |                    |                |                    |                |
| $x_6$ | $y_6$ | $\delta y_{11/2}$ |                |                    |                |                    |                |

Thus, we observe that all the odd differences have a fractional suffix and all the even differences with the same subscript lie horizontally.

The following alternative notation may also be adopted to introduce finite difference operators. Let  $y = f(x)$  be a functional relation between  $x$  and  $y$ , which is also denoted by  $y_x$ .

Suppose, we are given consecutive values of  $x$  differing by  $h$  say  $x, x+h, x+2h, x+3h$ , etc. The corresponding values of  $y$  are  $y_x, y_{x+h}, y_{x+2h}, y_{x+3h}$ ,

As before, we can form the differences of these values.

Thus

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly,

$$\nabla y_x = y_x - y_{x-h} = f(x) - f(x-h)$$

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

To be explicit, we write

$$\begin{aligned}
 \Delta y_0 &= y_1 - y_0 \\
 \Delta y_1 &= y_2 - y_1 \\
 &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 \Delta y_{n-1} &= y_n - y_{n-1}
 \end{aligned}$$

$$\nabla y_i = y_i - y_{i-1} \quad i = n, (n-1), \dots, 1$$

OR

$$\begin{aligned}
 \nabla y_1 &= y_1 - y_0 \\
 \nabla y_2 &= y_2 - y_1 \\
 &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 \nabla y_n &= y_n - y_{n-1}
 \end{aligned}$$

$$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1,$$

In General,

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

### Shift operator, E

Let  $y = f(x)$  be a function of  $x$ , and let  $x$  takes the consecutive values  $x, x + h, x + 2h$ , etc. We then define an operator having the property

$$E f(x) = f(x + h)$$

Thus, when  $E$  operates on  $f(x)$ , the result is the next value of the function. Here,  $E$  is called the shift operator. If we apply the operator  $E$  twice on  $f(x)$ , we get

$$\begin{aligned} E^2 f(x) &= E[E f(x)] \\ &= E[f(x + h)] = f(x + 2h) \end{aligned}$$

Thus, in general, if we apply the operator ' $E$ '  $n$  times on  $f(x)$ , we get

$$E^n f(x) = f(x + nh)$$

OR

$$E^n y_x = y_{x+nh}$$

$$E y_0 = y_1, \quad E^2 y_0 = y_2, \quad E^4 y_0 = y_4, \quad \dots, \quad E^2 y_2 = y_4$$

The inverse operator  $E^{-1}$  is defined as

$$E^{-1} f(x) = f(x - h)$$

Similarly

$$E^{-n} f(x) = f(x - nh)$$

### Average Operator, $\mu$

it is defined as

$$\begin{aligned} \mu f(x) &= \frac{1}{2} \left[ f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right] \\ &= \frac{1}{2} [y_{x+(h/2)} + y_{x-(h/2)}] \end{aligned}$$

### Differential Operator, $D$

it is defined as

$$\left. \begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned} \right\}$$

### Important Results Using $\{\Delta, \nabla, \delta, E, \mu\}$

$$\begin{aligned} \Delta y_x &= y_{x+h} - y_x = E y_x - y_x \\ &= (E - 1) y_x \\ \Rightarrow \Delta &= E - 1 \end{aligned}$$

Also

$$\begin{aligned} \nabla y_x &= y_x - y_{x-h} = y_x - E^{-1} y_x \\ &= (1 - E^{-1}) y_x \end{aligned}$$

$$\Rightarrow \nabla = 1 - E^{-1} = \frac{E - 1}{E}$$

And

$$\begin{aligned}\delta y_x &= y_{x+(h/2)} - y_{x-(h/2)} \\ &= E^{1/2} y_x - E^{-1/2} y_x \\ &= (E^{1/2} - E^{-1/2}) y_x\end{aligned}$$

$$\delta = E^{1/2} - E^{-1/2}$$

The definition of  $\mu$  and  $E$  similarly yields

$$\begin{aligned}\mu y_x &= \frac{1}{2} [y_{x+(h/2)} + y_{x-(h/2)}] \\ &= \frac{1}{2} (E^{1/2} + E^{-1/2}) y_x \\ \Rightarrow \mu &= \frac{1}{2} (E^{1/2} + E^{-1/2})\end{aligned}$$

We know that

$$\begin{aligned}E y_x &= y_{x+h} = f(x+h) \\ E y_x &= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots \\ &= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \dots \\ &= \left( 1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \dots \right) f(x) = e^{hD} y_x\end{aligned}$$

Thus

$$hD = \log E$$

**Example:**

Prove that

$$\begin{aligned}hD &= \log(1 + \Delta) \\ &= -\log(1 - \nabla) \\ &= \sinh^{-1}(\mu \delta)\end{aligned}$$

**Solution:**

Using the standard relations we have

$$\begin{aligned}hD &= \log E \\ &= \log(1 + \Delta) \\ &= -\log E^{-1} \\ &= -\log(1 - \nabla)\end{aligned}$$

$$\begin{aligned}
\mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) \\
&= \frac{1}{2}(E - E^{-1}) \\
&= \frac{1}{2}(e^{hD} - e^{-hD}) \\
&= \sinh(hD) \\
\Rightarrow hD &= \sinh^{-1} \mu\delta
\end{aligned}$$

### Example

Prove that

$$\begin{aligned}
1) \quad 1 + \delta^2 \mu^2 &= \left(1 + \frac{\delta^2}{2}\right)^2 \\
2) \quad E^{1/2} &= \mu + \frac{\delta}{2} \\
3) \quad \Delta &= \frac{\delta^2}{2} + \delta \sqrt{1 + (\delta^2/4)} \\
4) \quad \mu\delta &= \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2} \\
5) \quad \mu\delta &= \frac{\Delta + \nabla}{2}
\end{aligned}$$

### Solution

From the definition, we have:

$$\begin{aligned}
(1) \quad \mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) \\
\therefore 1 + \mu^2 \delta^2 &= 1 + \frac{1}{4}(E^2 - 2 + E^{-2}) = \frac{1}{4}(E + E^{-1})^2 \\
1 + \frac{\delta^2}{2} &= 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2 = \frac{1}{2}(E + E^{-1})^2
\end{aligned}$$

$$\begin{aligned}
(2) \quad \mu + (\delta/2) &= \frac{1}{2}(E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}) = E^{1/2}
\end{aligned}$$

(3)

$$\begin{aligned}
\frac{\delta^2}{2} + \delta \sqrt{1 + (\delta^2/4)} &= \frac{(E^{1/2} - E^{-1/2})^2}{2} + \frac{(E^{1/2} - E^{-1/2}) \sqrt{1 + \frac{1}{4}(E^{1/2} - E^{-1/2})^2}}{1} \\
&= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2}) \\
&= \frac{E - 2 + E^{-1}}{2} + \frac{E - E^{-1}}{2} \\
&= E - 1 = \Delta
\end{aligned}$$

(4)

$$\begin{aligned}
\mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) \\
&= \frac{1}{2}(1 + \Delta - E^{-1}) = \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1}) \\
&= \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E - 1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}
\end{aligned}$$

(5)

$$\begin{aligned}
\mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) \\
&= \frac{1}{2}(E - E^{-1}) \\
&= \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{1}{2}(\Delta + \nabla)
\end{aligned}$$

### Interpolation Newton's Forward Difference Formula

Let  $y = f(x)$  be a function which takes values  $f(x_0), f(x_0 + h), f(x_0 + 2h), \dots$ , corresponding to various equi-spaced values of  $x$  with spacing  $h$ , say  $x_0, x_0 + h, x_0 + 2h, \dots$ .

Suppose, we wish to evaluate the function  $f(x)$  for a value  $x_0 + ph$ , where  $p$  is any real number, then for any real number  $p$ , we have the operator  $E$  such that

$$\begin{aligned} E^p f(x) &= f(x + ph). \\ f(x_0 + ph) &= E^p f(x_0) = (1 + \Delta)^p f(x_0) \\ &= \left[ 1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots \right] f(x_0) \\ f(x_0 + ph) &= f(x_0) + p\Delta f(x_0) \\ &+ \frac{p(p-1)}{2!} \Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!} \Delta^3 f(x_0) \\ &+ \dots + \frac{p(p-1)\dots(p-n+1)}{n!} \Delta^n f(x_0) + \text{Error} \end{aligned}$$

This is known as Newton's forward difference formula for interpolation, which gives the value of  $f(x_0 + ph)$  in terms of  $f(x_0)$  and its leading differences.

This formula is also known as Newton-Gregory forward difference interpolation formula. Here  $p = (x - x_0)/h$ .

An alternate expression is

$$\begin{aligned} y_x &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \\ &+ \frac{p(p-1)(p-n+1)}{n!} \Delta^n y_0 + \text{Error} \end{aligned}$$

If we retain  $(r + 1)$  terms, we obtain a polynomial of degree  $r$  agreeing with  $y_x$  at

$x_0, x_1, \dots, x_r$ .

This formula is mainly used for interpolating the values of  $y$  near the beginning of a set of tabular values and for extrapolating values of  $y$ , a short distance backward from  $y_0$ .

### Example:

Evaluate  $f(15)$ , given the following table of values:

|            |    |    |    |    |     |
|------------|----|----|----|----|-----|
| $x$        | 10 | 20 | 30 | 40 | 50  |
| $y = f(x)$ | 46 | 66 | 81 | 93 | 101 |

### Solution:

The forward differences are tabulated as

| $x$ | $y = f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|-----|------------|------------|--------------|--------------|--------------|
| 10  | 46         |            |              |              |              |
| 20  | 66         | 20         |              |              |              |
| 30  | 81         | 15         | -5           |              |              |
| 40  | 93         | 12         | -3           | 2            |              |
| 50  | 101        | 8          | -4           | -1           | -3           |

We have Newton's forward difference interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots + \frac{p(p-1)(p-n+1)}{n!} \Delta^n y_0 + \text{Error}$$

Here we have

$$x_0 = 10, \quad y_0 = 46, \quad \Delta y_0 = 20, \\ \Delta^2 y_0 = -5, \quad \Delta^3 y_0 = 2, \quad \Delta^4 y_0 = -3$$

Let  $y_{15}$  be the value of  $y$  when  $x = 15$ , then

$$p = \frac{x - x_0}{h} = \frac{15 - 10}{10} = 0.5$$

$$f(15) = y_{15} = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2}(-5) \\ + \frac{(0.5)(0.5-1)(0.5-2)}{6}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(-3) \\ = 46 + 10 + 0.625 + 0.125 + 0.1172 \\ f(15) = 56.8672 \text{ correct to four decimal places.}$$

### Example

Find Newton's forward difference, interpolating polynomial for the following data:

| $x$        | 0.1 | 0.2  | 0.3  | 0.4 | 0.5  |
|------------|-----|------|------|-----|------|
| $y = f(x)$ | 1.4 | 1.56 | 1.76 | 2   | 2.28 |

**Solution;**

The forward difference table to the given data is

| $x$ | $y = f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|-----|------------|------------|--------------|--------------|--------------|
| 0.1 | 0.14       |            |              |              |              |
| 0.2 | 1.56       | 0.16       |              |              |              |
| 0.3 | 1.76       | 0.2        | 0.04         |              |              |
| 0.4 | 2          | 0.24       | 0.04         | 0            |              |
| 0.5 | 2.28       | 0.28       | 0.04         | 0            | 0            |

Since, 3<sup>rd</sup> and 4<sup>th</sup> leading differences are zero, we have Newton's forward difference interpolating formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0$$

In this problem,

$$x_0 = 0.1, y_0 = 1.40,$$

$$\Delta y_0 = 0.16, \Delta^2 y_0 = 0.04,$$

and

$$p = \frac{x-0.1}{0.1} = 10x-1$$

Substituting these values,

$$y = f(x) = 1.40 + (10x-1)(0.16) + \frac{(10x-1)(10x-2)}{2}(0.04)$$

This is the required Newton's interpolating polynomial.

### Example

Estimate the missing figure in the following table:

| $x$        | 1 | 2 | 3 | 4   | 5  |
|------------|---|---|---|-----|----|
| $y = f(x)$ | 2 | 5 | 7 | --- | 32 |

### Solution

Since we are given four entries in the table, the function  $y = f(x)$  can be represented by a polynomial of degree three.

$$\Delta^3 f(x) = \text{Constant}$$

$$\text{and } \Delta^4 f(x) = 0, \quad \forall x$$

In particular,

$$\Delta^4 f(x_0) = 0$$

Equivalently,

$$(E-1)^4 f(x_0) = 0$$

Expanding, we have

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)f(x_0) = 0$$

That is,

$$f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0) = 0$$

Using the values given in the table, we obtain

$$32 - 4f(x_3) + 6 \times 7 - 4 \times 5 + 2 = 0$$

which gives  $f(x_3)$ , the missing value equal to 14.

### Example

Consider the following table of values

|      |       |       |       |       |       |
|------|-------|-------|-------|-------|-------|
| x    | .2    | .3    | .4    | .5    | .6    |
| F(x) | .2304 | .2788 | .3222 | .3617 | .3979 |

Find  $f(.36)$  using Newton's Forward Difference Formula.

**Solution**

| x   | y = f(x) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|-----|----------|------------|--------------|--------------|--------------|
| 0.2 | 0.2304   | 0.0484     | -0.005       | 0.0011       | -0.0005      |
| 0.3 | 0.2788   | 0.0434     | -0.0039      | 0.0006       |              |
| 0.4 | 0.3222   | 0.0395     | -0.0033      |              |              |
| 0.5 | 0.3617   | 0.0362     |              |              |              |
| 0.6 | 0.3979   |            |              |              |              |

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots + \frac{p(p-1)(p-2)\dots(p-n+1)}{n!}\Delta^n y_0$$

Where

$$x_0 = 0.2, \quad y_0 = 0.2304, \quad \Delta y_0 = 0.0484, \\ \Delta^2 y_0 = -0.005, \quad \Delta^3 y_0 = 0.0011, \quad \Delta^4 y_0 = -0.0005 \quad p = \frac{x - x_0}{h} = \frac{0.36 - 0.2}{0.1} = 1.6$$

$$\begin{aligned} y_x &= 0.2304 + 1.6(0.0484) + \frac{1.6(1.6-1)}{2!}(-0.005) + \frac{1.6(1.6-1)(1.6-2)}{3!}(0.0011) + \frac{1.6(1.6-1)(1.6-2)(1.6-3)}{4!}(-0.0005) \\ &= 0.2304 + 0.077441 - 0.0024 + \frac{1.6(6)(-.4)}{6}(0.0011) + \frac{1.6(6)(-.4)(-1.4)}{24}(-0.0005) \\ &= 0.3078 - 0.0024 - 0.00007 - 0.00001 \\ &= 0.3053 \end{aligned}$$

**Example**

Find a cubic polynomial in  $x$  which takes on the values

-3, 3, 11, 27, 57 and 107, when  $x = 0, 1, 2, 3, 4$  and 5 respectively.

**Solution**

Here, the observations are given at equal intervals of unit width.

To determine the required polynomial, we first construct the difference table

**Difference Table**

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|-----|--------|---------------|-----------------|-----------------|
| 0   | -3     | 6             |                 |                 |
| 1   | 3      | 8             | 2               |                 |
| 2   | 11     | 16            | 8               | 6               |
| 3   | 27     | 30            | 14              | 6               |
| 4   | 57     | 30            | 20              | 6               |
| 5   | 107    | 50            |                 |                 |

Since the 4<sup>th</sup> and higher order differences are zero, the required Newton's interpolation formula

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0)$$

Here

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

$$\Delta f(x_0) = 6$$

$$\Delta^2 f(x_0) = 2$$

$$\Delta^3 f(x_0) = 6$$

Substituting these values into the formula, we have

$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

$$f(x) = x^3 - 2x^2 + 7x - 3,$$

The required cubic polynomial.

## NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

For interpolating the value of the function  $y = f(x)$  near the end of table of values, and to extrapolate value of the function a short distance forward from  $y_n$ , Newton's backward interpolation formula is used

### Derivation

Let  $y = f(x)$  be a function which takes on values  $f(x_n), f(x_{n-h}), f(x_{n-2h}), \dots, f(x_0)$  corresponding to equispaced values  $x_n, x_{n-h}, x_{n-2h}, \dots, x_0$ . Suppose, we wish to evaluate the function  $f(x)$  at  $(x_n + ph)$ , where  $p$  is any real

number, then we have the shift operator  $E$ , such that

$$f(x_n + ph) = E^p f(x_n) = (E^{-1})^{-p} f(x_n) = (1 - \nabla)^{-p} f(x_n)$$

Binomial expansion yields,

$$f(x_n + ph) = \left[ 1 + p\nabla + \frac{p(p+1)}{2!} \nabla^2 + \frac{p(p+1)(p+2)}{3!} \nabla^3 + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n + \text{Error} \right] f(x_n)$$

That is

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!} \nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{3!} \nabla^3 f(x_n) + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n f(x_n) + \text{Error}$$

This formula is known as Newton's backward interpolation formula. This formula is also known as Newton's-Gregory backward difference interpolation formula.

If we retain  $(r+1)$  terms, we obtain a polynomial of degree  $r$  agreeing with  $f(x)$  at  $x_n$ ,

$x_{n-1}, \dots, x_{n-r}$ . Alternatively, this formula can also be written as

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n y_n + \text{Error}$$

Here  $p = \frac{x - x_n}{h}$

### Example

For the following table of values, estimate  $f(7.5)$ .

| $x$        | 1 | 2 | 3  | 4  | 5   | 6   | 7   | 8   |
|------------|---|---|----|----|-----|-----|-----|-----|
| $y = f(x)$ | 1 | 8 | 27 | 64 | 125 | 216 | 343 | 512 |

**Solution**

The value to be interpolated is at the end of the table. Hence, it is appropriate to use Newton's backward interpolation formula. Let us first construct the backward difference table for the given data

**Difference Table**

| $x$ | $y = f(x)$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|-----|------------|------------|--------------|--------------|--------------|
| 1   | 1          |            |              |              |              |
| 2   | 8          | 7          |              |              |              |
| 3   | 27         | 19         | 12           |              |              |
| 4   | 64         | 37         | 18           | 6            |              |
| 5   | 125        | 61         | 24           | 6            | 0            |
| 6   | 216        | 91         | 30           | 6            | 0            |
| 7   | 343        | 127        | 36           | 6            | 0            |
| 8   | 512        | 169        | 42           | 6            | 0            |

<sup>th</sup> Since the 4 and higher order differences are zero, the required Newton's backward interpolation formula is

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

In this problem,

$$\begin{aligned}
 p &= \frac{x - x_n}{h} = \frac{7.5 - 8.0}{1} = -0.5 \\
 \nabla y_n &= 169, \quad \nabla^2 y_n = 42, \quad \nabla^3 y_n = 6 \\
 y_{7.5} &= 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2}(42) \\
 &\quad + \frac{(-0.5)(0.5)(1.5)}{6}(6) \\
 &= 512 - 84.5 - 5.25 - 0.375 \\
 &= 421.875
 \end{aligned}$$

**Example**

The sales for the last five years is given in the table below. Estimate the sales for the year 1979

| Year             | 1974 | 1976 | 1978 | 1980 | 1982 |
|------------------|------|------|------|------|------|
| Sales (in lakhs) | 40   | 43   | 48   | 52   | 57   |

**Solution****Newton's backward difference Table**

| $x$  | $y$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|------|-----|------------|--------------|--------------|--------------|
| 1974 | 40  |            |              |              |              |
| 1976 | 43  | 3          |              |              |              |
| 1978 | 48  | 5          | 2            |              |              |
| 1980 | 52  | 4          | -1           | 3            |              |
| 1982 | 57  | 5          | 1            | 2            | 5            |

In this example,

$$p = \frac{1979-1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1,$$

$$\nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

Newton's interpolation formula gives

$$\begin{aligned}
 y_{1979} &= 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1) + \frac{(-1.5)(-0.5)(0.5)}{6}(2) \\
 &\quad + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5) \\
 &= 57 - 7.5 + 0.375 + 0.125 + 0.1172
 \end{aligned}$$

Therefore,

$$y_{1979} = 50.1172$$

**Example**

Consider the following table of values

|      |   |     |     |     |     |     |
|------|---|-----|-----|-----|-----|-----|
| x    | 1 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
| F(x) | 2 | 2.1 | 2.3 | 2.7 | 3.5 | 4.5 |

Use Newton's Backward Difference Formula to estimate the value of  $f(1.45)$ .

**Solution**

| x   | y=F(x) | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ | $\nabla^5 y$ |
|-----|--------|------------|--------------|--------------|--------------|--------------|
| 1   | 2      |            |              |              |              |              |
| 1.1 | 2.1    | 0.1        |              |              |              |              |
| 1.2 | 2.3    | 0.2        | 0.1          |              |              |              |
| 1.3 | 2.7    | 0.4        | 0.2          | 0.1          |              |              |
| 1.4 | 3.5    | 0.8        | 0.4          | 0.2          | 0.1          |              |
| 1.5 | 4.5    | 1          | 0.2          | -0.2         | -0.4         | -0.5         |

$$p = \frac{x - x_n}{h} = \frac{1.45 - 1.5}{0.1} = -0.5, \nabla y_n = 1, \nabla^2 y_n = .2, \nabla^3 y_n = -.2, \nabla^4 y_n = -.4,$$

$$\nabla^5 y_n = -.5$$

As we know that

$$\begin{aligned} y_x &= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \\ &\quad + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n \\ y_x &= 4.5 + (-0.5)(1) + \frac{(-0.5)(-0.5+1)}{2!}(0.2) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(-0.2) \\ &\quad + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(-0.4) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)}{5!}(-0.5) \end{aligned}$$

$$= 4.5 - 0.5 - 0.025 + 0.0125 + 0.015625 + 0.068359$$

$$= 4.07148$$

### LAGRANGE'S INTERPOLATION FORMULA

Newton's interpolation formulae developed earlier can be used only when the values of the independent variable  $x$  are equally spaced. Also the differences of  $y$  must ultimately become small.

If the values of the independent variable are not given at equidistant intervals, then we have the basic formula associated with the name of Lagrange which will be derived now.

Let  $y = f(x)$  be a function which takes the values,  $y_0, y_1, \dots, y_n$  corresponding to  $x_0, x_1, \dots, x_n$ . Since there are  $(n + 1)$  values of  $y$  corresponding to  $(n + 1)$  values of  $x$ , we

can represent the function  $f(x)$  by a polynomial of degree  $n$ .

Suppose we write this polynomial in the form,

$$f(x) = A_0 x^n + A_1 x^{n-1} + \dots + A_n$$

or in the form,

$$\begin{aligned} y = f(x) &= a_0(x - x_1)(x - x_2) \cdots (x - x_n) \\ &+ a_1(x - x_0)(x - x_2) \cdots (x - x_n) \\ &+ a_2(x - x_0)(x - x_1) \cdots (x - x_n) + \cdots \\ &+ a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \end{aligned}$$

Here, the coefficients  $a_k$  are so chosen as to satisfy this equation by the  $(n + 1)$  pairs  $(x_i, y_i)$ . Thus we get

$$y_0 = f(x_0) = a_0(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)$$

Therefore,

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

Similarly, we obtain

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

and

$$\begin{aligned} a_i &= \frac{y_i}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} \\ a_n &= \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} \end{aligned}$$

Substituting the values of  $a_0, a_1, \dots, a_n$  we get

$$\begin{aligned} y = f(x) &= \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} y_1 + \\ &+ \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} y_i + \cdots \end{aligned}$$

The Lagrange's formula for interpolation

This formula can be used whether the values  $x_0, x_1, \dots, x_n$  are equally spaced or not.

Alternatively, this can also be written in compact form as

$$\begin{aligned} y = f(x) &= L_0(x)y_0 + L_1(x_1)y_1 + L_i(x_i)y_i + \dots + L_n(x_n)y_n \\ &= \sum_{k=0}^n L_k(x)y_k = \sum_{k=0}^n L_k(x)f(x_k) \end{aligned}$$

Where

$$L_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}$$

We can easily observe that,

$$L_i(x_i) = 1 \quad \text{and} \quad L_i(x_j) = 0, i \neq j.$$

Thus introducing *Kronecker delta* notation

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Further, if we introduce the notation

$$\Pi(x) = \prod_{i=0}^n (x - x_i) = (x - x_0)(x - x_1)\dots(x - x_n)$$

That is  $\Pi(x)$  is a product of  $(n + 1)$  factors. Clearly, its derivative  $\Pi'(x)$  contains a sum of  $(n + 1)$  terms in each of which one of the factors of  $\Pi(x)$  will be absent.

We also define,

$$P_k(x) = \prod_{i \neq k} (x - x_i)$$

Which is same as  $\Pi(x)$  except that the factor  $(x - x_k)$  is absent. Then

$$\Pi'(x) = P_0(x) + P_1(x) + \dots + P_n(x)$$

But, when  $x = x_k$ , all terms in the above sum vanishes except  $P_k(x_k)$

Hence

$$\begin{aligned} \Pi'(x_k) &= P_k(x_k) = (x_k - x_0)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n) \\ L_k(x) &= \frac{P_k(x)}{P_k(x_k)} = \frac{P_k(x)}{\Pi'(x_k)} \\ &= \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} \end{aligned}$$

Finally, the Lagrange's interpolation polynomial of degree  $n$  can be written as

$$\begin{aligned} y(x) = f(x) &= \sum_{k=0}^n \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} f(x_k) \\ &= \sum_{k=0}^n L_k(x)f(x_k) = \sum_{k=0}^n L_k(x)y_k \end{aligned}$$

**Example**

Find Lagrange's interpolation polynomial fitting the points

$$y(1) = -3, y(3) = 0,$$

$$y(4) = 30, y(6) = 132. \text{ Hence find } y(5).$$

**Solution**

The given data can be arranged as

|            |    |   |    |     |
|------------|----|---|----|-----|
| $x$        | 1  | 3 | 4  | 6   |
| $y = f(x)$ | -3 | 0 | 30 | 132 |

Using Lagrange's interpolation formula, we have

$$y(x) = f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0) \\ + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

On simplification, we get

$$y(x) = \frac{1}{10}(-5x^3 + 135x^2 - 460x + 300) \\ = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$$

Which is required Lagrange's interpolation polynomial.

Now,  $y(5) = 75$ .

**Example**

Given the following data, evaluate  $f(3)$  using Lagrange's interpolating polynomial.

|        |   |   |    |
|--------|---|---|----|
| $x$    | 1 | 2 | 5  |
| $f(x)$ | 1 | 4 | 10 |

**Solution**

Using Lagrange's formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Therefore,

$$f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1) + \frac{(3-1)(3-5)}{(2-1)(2-5)}(4) + \frac{(3-1)(3-2)}{(5-1)(5-2)}(10) \\ = 6.49999 = 6.5$$

**Example**

Find the interpolating polynomial for the data using Lagrange's formula

|      |   |    |    |
|------|---|----|----|
| x    | 1 | 2  | -4 |
| F(x) | 3 | -5 | -4 |

**Solution:**

As Lagrange's formula for interpolating polynomial is given by,

$$\begin{aligned}
 y(x) = f(x) &= \frac{(x-x_1)(x-x_2)}{(x-x_0)(x-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x-x_1)(x-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x-x_2)(x-x_1)} f(x_2) \\
 &= \frac{(x-2)(x+4)}{(1-2)(1+4)} (3) + \frac{(x-1)(x+4)}{(2-1)(2+4)} (-5) + \frac{(x-1)(x-2)}{(-4-1)(-4-2)} (-4) \\
 &= -\frac{3}{5} (x^2 + 2x - 8) - \frac{5}{6} (x^2 + 3x - 4) - \frac{4}{30} (x^2 - 3x + 2) \\
 &= -\frac{3}{5} x^2 - \frac{6}{5} x + \frac{24}{5} - \frac{5}{6} x^2 - \frac{15}{6} x + \frac{20}{6} - \frac{4}{30} x^2 + \frac{12}{30} x - \frac{8}{30} \\
 &= \left( -\frac{3}{5} - \frac{5}{6} - \frac{4}{30} \right) x^2 + \left( -\frac{6}{5} - \frac{15}{6} + \frac{12}{30} \right) x + \left( \frac{24}{5} + \frac{20}{6} - \frac{8}{30} \right) \\
 &= -\frac{47}{30} x^2 - \frac{33}{10} x + \frac{118}{15}
 \end{aligned}$$

Which is the required polynomial.

### DIVIDED DIFFERENCES

Let us assume that the function  $y = f(x)$  is known for several values of  $x$ ,  $(x_i, y_i)$ , for

$i=0,1,\dots,n$ . The divided differences of orders 0, 1, 2, ...,  $n$  are now defined recursively as:

$$y[x_0] = y(x_0) = y_0$$

is the zero-th order divided difference

The first order divided difference is defined as

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, the higher order divided differences are defined in terms of lower order divided differences by the relations of the form

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

Generally,

$$y[x_0, x_1, \dots, x_n] = \frac{1}{x_n - x_0} [y[x_1, x_2, \dots, x_n] - y[x_0, x_1, \dots, x_{n-1}]]$$

### Standard format of the Divided Differences

| $x$   | $y(x)$ | 1 <sup>st</sup> order | 2 <sup>nd</sup> order | 3 <sup>rd</sup> order   | 4 <sup>th</sup> order        |
|-------|--------|-----------------------|-----------------------|-------------------------|------------------------------|
| $x_0$ | $y_0$  |                       |                       |                         |                              |
| $x_1$ | $y_1$  | $y[x_1, x_0]$         |                       |                         |                              |
| $x_2$ | $y_2$  | $y[x_2, x_1]$         | $y[x_0, x_1, x_2]$    | $y[x_0, x_1, x_2, x_3]$ |                              |
| $x_3$ | $y_3$  | $y[x_3, x_2]$         | $y[x_1, x_2, x_3]$    | $y[x_1, x_2, x_3, x_4]$ | $y[x_0, x_1, x_2, x_3, x_4]$ |
| $x_4$ | $y_4$  | $y[x_4, x_3]$         | $y[x_2, x_3, x_4]$    |                         |                              |

We can easily verify that the divided difference is a symmetric function of its arguments. That is,

$$y[x_1, x_0] = y[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

Now

$$= \frac{1}{x_2 - x_0} \left( \frac{y_1 - y_0}{x_2 - x_1} - \frac{y_0 - y_1}{x_1 - x_0} \right)$$

Therefore

$$y[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

This is symmetric form, hence suggests the general result as

$$y[x_0, \dots, x_k] = \frac{y_0}{(x_0 - x_1) \cdots (x_0 - x_k)} + \frac{y_1}{(x_1 - x_0) \cdots (x_1 - x_k)} + \cdots + \frac{y_k}{(x_k - x_0) \cdots (x_k - x_{k-1})}$$

OR

$$y[x_0, \dots, x_k] = \sum_{i=0}^k \frac{y_i}{\prod_{\substack{j=0 \\ j \neq i}}^k (x_i - x_j)}$$

Example:

Construct the Newton's divided difference table for values of  $x=1,2,3,4,5,6$  and  $f(x) = -3, 0, 15, 48, 105, 192$ .

Solution:

| <b>x</b> | <b>F (x)</b> | <b>1<sup>st</sup><br/>difference</b> | <b>2<sup>nd</sup><br/>difference</b> | <b>3<sup>rd</sup><br/>difference</b> |
|----------|--------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 1        | -3           |                                      | 15-3/3-1=6                           |                                      |
| 2        | 0            | 0 - (-3)=3                           | 33-15/4-2=9                          | 9-6/4-1=1                            |
| 3        | 15           | 15-0=15                              | 57-33/5-3=12                         | 12-9/5-2=1                           |
| 4        | 48           | 48-15=33                             | 87-57/6-4=15                         | 15-12/6-3=1                          |
| 5        | 105          | 105-48=57                            |                                      |                                      |
| 6        | 192          | 192-105=87                           |                                      |                                      |

### NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA

Let  $y = f(x)$  be a function which takes values  $y_0, y_1, \dots, y_n$  corresponding to  $x = x_i, i = 0, 1, \dots, n$ . We choose an interpolating polynomial, interpolating at  $x = x_i, i = 0, 1, \dots, n$  in the following form

$$y = f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

Here, the coefficients  $a_k$  are so chosen as to satisfy above equation by the  $(n + 1)$  pairs  $(x_i, y_i)$ .

Thus, we have

$$\left. \begin{aligned} y(x_0) &= f(x_0) = y_0 = a_0 \\ y(x_1) &= f(x_1) = y_1 = a_0 + a_1(x_1 - x_0) \\ y(x_2) &= f(x_2) = y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ &\dots \\ y(x_n) &= a_0 + a_1(x_n - x_0) + a_2(x_n - x_0)(x_n - x_1) + \dots + a_n(x_n - x_0) \dots (x_n - x_{n-1}) \end{aligned} \right\}$$

The coefficients  $a_0, a_1, \dots, a_n$  can be easily obtained from the above system of equations, as they form a lower triangular matrix. The first equation gives

$$a_0 = y(x_0) = y_0$$

The second equation gives

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = y[x_0, x_1]$$

Third equation yields

$$a_2 = \frac{y_2 - y_0 - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

Which can be rewritten as

$$a_2 = \frac{\left[ y_2 - y_0 + \left( \frac{y_1 - y_0}{x_1 - x_0} \right) (x_2 - x_0) \right] - (x_2 - x_0)y[x_0, x_1]}{(x_2 - x_0)(x_2 - x_1)}$$

That is

$$a_2 = \frac{y_2 - y_1 + y[x_0, x_1](x_1 - x_2)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

Thus, in terms of second order divided differences, we have

$$a_2 = y[x_0, x_1, x_2]$$

Similarly, we can show that

Newton's divided difference interpolation formula can be written as

$$y = f(x) = y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2] + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1})y[x_0, x_1, \dots, x_n]$$

Newton's divided differences can also be expressed in terms of forward, backward and central differences. They can be easily derived.

Assuming equi-spaced values of abscissa, we have

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}}{2h} = \frac{\Delta^2 y_0}{2!h^2}$$

By induction, we can in general arrive at the result

$$y[x_0, x_1, \dots, x_n] = \frac{\Delta^n y_0}{n!h^n}$$

Similarly,

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\nabla y_1}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\nabla y_2}{h} - \frac{\nabla y_1}{h}}{2h} = \frac{\nabla^2 y_2}{2!h^2}$$

In General, we have

$$y[x_0, x_1, \dots, x_n] = \frac{\nabla^n y_n}{n!h^n}$$

Also, in terms of central differences, we have

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\delta y_{1/2}}{h}$$

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\delta y_{3/2}}{h} - \frac{\delta y_{1/2}}{h}}{2h} = \frac{\delta^2 y_{1/2}}{2!h^2}$$

In general, we have the following pattern

$$\left. \begin{aligned} y[x_0, x_1, \dots, x_m] &= \frac{\delta^{2m} y_m}{(2m)!h^{2m}} \\ y[x_0, x_1, \dots, x_{2m+1}] &= \frac{\delta^{2m+1} y_{m+(1/2)}}{(2m+1)!h^{2m+1}} \end{aligned} \right\}$$

**Example**

Find the interpolating polynomial by (i) Lagrange's formula and (ii) Newton's divided difference formula for the following data. Hence show that they

|   |   |   |   |   |
|---|---|---|---|---|
| x | 0 | 1 | 2 | 4 |
| y | 1 | 1 | 2 | 5 |

represent the same interpolating polynomial.

| X | Y | 1 <sup>st</sup> D.D | 2 <sup>nd</sup> D.D | 3 <sup>rd</sup> D.D |
|---|---|---------------------|---------------------|---------------------|
| 0 | 1 |                     |                     |                     |
| 1 | 1 | 0                   |                     |                     |
| 2 | 2 | 1                   | 1/2                 | -1/2                |
| 4 | 5 | 3/2                 | 1/6                 |                     |

**Solution** The divided difference table for the given data is constructed as follows:

**i) Lagrange's interpolation formula gives**

$$\begin{aligned}
 y = f(x) &= \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(1) \\
 &+ \frac{(x-0)(x-1)(x-4)}{(2)(2-1)(2-4)}(2) + \frac{(x-0)(x-1)(x-2)}{4(4-1)(4-2)}(5) \\
 &= -\frac{(x^3 - 7x^2 + 14x - 8)}{8} + \frac{x^3 - 6x^2 + 8x}{3} - \frac{x^3 - 5x^2 + 4x}{2} \\
 &+ \frac{5(x^3 - 3x^2 + 2x)}{24} \\
 &= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1
 \end{aligned}$$

**(ii) Newton's divided difference formula gives**

$$\begin{aligned}
 y = f(x) &= 1 + (x-0)(0) + (x-0)(x-1)\left(\frac{1}{2}\right) + (x-0)(x-1)(x-2)\left(-\frac{1}{12}\right) \\
 &= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1
 \end{aligned}$$

We observe that the interpolating polynomial by both Lagrange's and Newton's divided difference formulae is one and the same.

**Note!**

Newton's formula involves less number of arithmetic operations than that of Lagrange's.

**Example**

Using Newton's divided difference formula, find the quadratic equation for the following

|   |   |   |   |
|---|---|---|---|
| X | 0 | 1 | 4 |
| Y | 2 | 1 | 4 |

data.

Hence find  $y(2)$ .

**Solution:**

The divided difference table for the given data is constructed as:

| X | Y | 1 <sup>st</sup> D.D | 2 <sup>nd</sup> D.D |
|---|---|---------------------|---------------------|
| 0 | 2 |                     |                     |
| 1 | 1 | -1                  | 1/2                 |
| 4 | 4 | 1                   |                     |

Now, using Newton's divided difference formula, we have

$$y = 2 + (x - 0)(-1) + (x - 0)(x - 1)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}(x^2 - 3x + 4)$$

Hence,

$$y(2) = 1.$$

**Example**

Find the equation of a cubic curve which passes through the points (4, -43), (7, 83), (9, 327) and (12, 1053) using Dividing Difference Formula.

**Solution**

The Newton's divided difference table is given by

| X  | Y    | 1 <sup>st</sup> divided difference | 2 <sup>nd</sup> divided difference | 3 <sup>rd</sup> divided difference |
|----|------|------------------------------------|------------------------------------|------------------------------------|
| 4  | -43  |                                    |                                    |                                    |
| 7  | 83   | 42                                 |                                    |                                    |
| 9  | 327  | 122                                | 16                                 |                                    |
| 12 | 1053 | 242                                | 24                                 | 1                                  |

*Newton's Divided Difference formulais*

$$\begin{aligned}
 y = f(x) &= y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2] + \cdots + \\
 &\quad (x - x_0)(x - x_1)\cdots(x - x_{n-1})y[x_0, x_1, \dots, x_n] \\
 &= -43 + (x - 4)(42) + (x - 4)(x - 7)(16) + (x - 4)(x - 7)(x - 9)(1) \\
 &= -43 + (x - 4)\{42 + 16x - 112 + x^2 - 16x + 63\} \\
 &= -43 + (x - 4)(x^2 - 7) \\
 &= -43 + x^3 - 7x - 4x^2 + 28 \\
 &= x^3 - 4x^2 - 7x - 15
 \end{aligned}$$

Which is the required polynomial.

### Example

A function  $y = f(x)$  is given at the sample points  $x = x_0, x_1$  and  $x_2$ . Show that the Newton's divided difference interpolation formula and the corresponding Lagrange's interpolation formula are identical.

### Solution

For the function  $y = f(x)$ , we have the data  $(x_i, y_i), i = 0, 1, 2$ .

The interpolation polynomial using Newton's divided difference formula is given as

$$\begin{aligned}
 y = f(x) &= y_0 + (x - x_0)y[x_0, x_1] \\
 &\quad + (x - x_0)(x - x_1)y[x_0, x_1, x_2]
 \end{aligned}$$

Using the definition of divided differences, we can rewrite the equation in the form

$$\begin{aligned}
 y &= y_0 + (x - x_0) \frac{y_1 - y_0}{x_1 - x_0} + (x - x_0)(x - x_1) \left[ \frac{y_2 - y_0}{(x_2 - x_0)(x_2 - x_1)} \right. \\
 &\quad \left. + \frac{y_1}{(x_1 - x_0)(x_1 - x_1)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} \right] \\
 &= \left[ 1 - \frac{(x - x_0)(x - x_1)}{(x_0 - x_1)(x_0 - x_2)} \right] y_0 \\
 &\quad + \left[ \frac{(x - x_0)}{(x_1 - x_0)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)} \right] y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2
 \end{aligned}$$

On simplification, it reduces to

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

which is the Lagrange's form of interpolation polynomial.

Hence two forms are identical.

### Newton's Divided Difference Formula with Error Term

Following the basic definition of divided differences, we have for any  $x$

$$\left. \begin{aligned} y(x) &= y_0 + (x - x_0)y[x, x_0] \\ y[x, x_0] &= y[x_0, x_1] + (x - x_1)y[x, x_0, x_1] \\ y[x, x_0, x_1] &= y[x_0, x_1, x_2] + (x - x_2)y[x, x_0, x_1, x_2] \\ &\vdots \\ y[x, x_0, \dots, x_{n-1}] &= y[x_0, x_1, \dots, x_n] + (x - x_n)y[x, x_0, \dots, x_n] \end{aligned} \right\}$$

Multiplying the second Equation by  $(x - x_0)$ , third by  $(x - x_0)(x - x_1)$  and so on, and the last by

$(x - x_0)(x - x_1) \dots (x - x_{n-1})$  and adding the resulting equations, we obtain

$$y(x) = y_0 + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2] + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1})y[x_0, x_1, \dots, x_n] + \varepsilon(x)$$

Where

Please note that for  $x = x_0, x_1, \dots, x_n$ , the error term  $\varepsilon(x)$  vanishes

### Error Term in Interpolation Formulae

We know, if  $y(x)$  is approximated by a polynomial  $P_n(x)$  of degree  $n$  then the error is given by

$$\varepsilon(x) = y(x) - P_n(x),$$

Where

$$\varepsilon(x) = (x - x_0)(x - x_1) \cdots (x - x_n) y[x, x_0, \dots, x_n]$$

Alternatively it is also expressed as

$$\varepsilon(x) = \Pi(x) y[x, x_0, \dots, x_n] = K \Pi(x)$$

Now, consider a function  $F(x)$ , such that

$$F(x) = y(x) - P_n(x) - K \Pi(x)$$

Determine the constant  $K$  in such a way that  $F(x)$  vanishes for  $x = x_0, x_1, \dots, x_n$  and also for an arbitrarily chosen point  $\bar{x}$ , which is different from the given  $(n + 1)$  points.

Let  $I$  denotes the closed interval spanned by the values  $x_0, \dots, x_n, \bar{x}$ . Then  $F(x)$  vanishes  $(n + 2)$  times in the interval  $I$ .

By Rolle's theorem  $F'(x)$  vanishes at least  $(n + 1)$  times in the interval  $I$ ,  $F''(x)$  at least  $n$  times, and so on.

Eventually, we can show that  $F^{(n+1)}(x)$  vanishes at least once in the interval  $I$ , say at  $x = \xi$ . Thus, we obtain

$$0 = y^{(n+1)}(\xi) - P_n^{(n+1)}(\xi) - K \Pi^{(n+1)}(\xi)$$

Since  $P_n(x)$  is a polynomial of degree  $n$ , its  $(n + 1)$ th derivative is zero. Also, from the definition of  $\Pi(x)$

$$\Pi^{(n+1)}(x) = (n+1)!.$$

Therefore we get

$$K = \frac{y^{(n+1)}(\xi)}{(n+1)!}$$

Hence

$$\varepsilon(x) = y(x) - P_n(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!} \Pi(x)$$

for some in the interval  $I$ .

Thus the error committed in replacing  $y(x)$  by either Newton's divided difference formula or by an identical Lagrange's formula is

$$\varepsilon(x) = \Pi(x) y[x, x_0, \dots, x_n] = \Pi(x) \frac{y^{(n+1)}(\xi)}{(n+1)!}$$

## INTERPOLATION IN TWO DIMENSIONS

Let  $u$  be a polynomial function in two variables, say  $x$  and  $y$ , in particular quadratic in  $x$  and cubic in  $y$ , which in general can be written as

$$u = f(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 y^3 + a_7 y^2 x + a_8 yx^2 + a_9 y^3 x + a_{10} y^2 x^2 + a_{11} y^3 x^2$$

This relation involves many terms. If we have to write a relation involving three or more variables, even low degree polynomials give rise to long expressions. If necessary, we can certainly write, but such complications can be avoided by handling each variable separately.

If we let  $x$ , a constant, say  $x = c$ , the equation simplifies to the form

$$u = \left|_{x=c} = b_0 + b_1 y + b_2 y^2 + b_3 y^3 \right.$$

Now we adopt the following procedure to interpolate at a point  $(1, m)$  in a table of two variables, by treating one variable a constant say  $x = x_1$ . The problem reduces to that of a single variable interpolation.

Any one of the methods discussed in preceding sections can then be applied to get  $f(x_1, m)$ . Then we repeat this procedure for various values of  $x$  say  $x = x_2, x_3, \dots, x_n$  keeping  $y$  constant. Thus, we get a new table with  $y$  constant at the value  $y = m$  and with  $x$  varying. We can then interpolate at  $x = 1$ .

**Example**

Tabulate the values of the function  $f(x) = x^2 + y^2 - y$  for  $x = 0, 1, 2, 3, 4$  and  $y = 0, 1, 2, 3, 4$ .

Using the table of the values, compute  $f(2.5, 3.5)$  by numerical double interpolation

**Solution**

The values of the function for the given values of the given values of  $x$  and  $y$  are given in

| X | ..... | Y | ..... |   |    |
|---|-------|---|-------|---|----|
|   | 0     | 1 | 2     | 3 | 4  |
| 0 | 0     | 0 | 2     | 6 | 12 |

the following table

|   |    |    |    |    |    |
|---|----|----|----|----|----|
| 1 | 1  | 1  | 3  | 7  | 13 |
| 2 | 4  | 4  | 6  | 10 | 16 |
| 3 | 9  | 9  | 11 | 15 | 21 |
| 4 | 16 | 16 | 18 | 22 | 28 |

Cont! Using quadratic interpolation in both x and y directions we need to consider three points in x and y directions. To start with we have to treat one variable as constant, say x. Keeping  $x = 2.5$ ,  $y = 3.5$  as near the center of the set, we choose the table of values corresponding to  $x = 1, 2, 3$  and  $y = 2, 3, 4$ .

Cont! The region of fit for the construction of our interpolation polynomial is shown in different color in the table

| X | ..... | Y  | ..... |    |    |
|---|-------|----|-------|----|----|
|   | 0     | 1  | 2     | 3  | 4  |
| 0 | 0     | 0  | 2     | 6  | 12 |
| 1 | 1     | 1  | 3     | 7  | 13 |
| 2 | 4     | 4  | 6     | 10 | 16 |
| 3 | 9     | 9  | 11    | 15 | 21 |
| 4 | 16    | 16 | 18    | 22 | 28 |

Cont !

Thus using Newton's forward difference formula, we have

|   | At $x=1$ |            |              |
|---|----------|------------|--------------|
| Y | f        | $\Delta f$ | $\Delta^2 f$ |
| 2 | 3        |            |              |
| 3 | 7        | 4          | 2            |
| 4 | 13       | 6          |              |

Cont !

Similarly

|   | At $x=2$ |            |              |
|---|----------|------------|--------------|
| Y | f        | $\Delta f$ | $\Delta^2 f$ |

|   |    |   |   |
|---|----|---|---|
| 2 | 6  |   |   |
| 3 | 10 | 4 | 2 |
| 4 | 16 | 6 |   |

Cont !  
Similarly

|   |        |            |              |
|---|--------|------------|--------------|
|   | At x=3 |            |              |
| Y | f      | $\Delta f$ | $\Delta^2 f$ |
| 2 | 11     |            |              |
| 3 | 15     | 4          | 2            |
| 4 | 21     | 6          |              |

$$\text{with } p = \frac{y - y_0}{h} = \frac{3.5 - 2}{1} = 1.5$$

Cont !

$$\text{with } p = \frac{y - y_0}{h} = \frac{3.5 - 2}{1} = 1.5$$

$$\begin{aligned} f(1, 3.5) &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!} \Delta^2 f_0 \\ &= 3 + (1.5)(4) + \frac{(1.5)(0.5)}{2} (2) = 9.75 \end{aligned}$$

Cont !

$$\begin{aligned} f(2, 3.5) &= 6 + (1.5)(4) + \frac{(1.5)(0.5)}{2} (2) = 12.75 \\ f(3, 3.5) &= 11 + (1.5)(4) + \frac{(1.5)(0.5)}{2} (2) = 17.75 \end{aligned}$$

Cont !

Therefore we arrive at the following result

|   |          |            |              |
|---|----------|------------|--------------|
|   | At x=3.5 |            |              |
| Y | f        | $\Delta f$ | $\Delta^2 f$ |
| 1 | 9.75     |            |              |
| 2 | 12.75    | 3          | 2            |
| 3 | 17.75    | 5          |              |

Cont !

$$\text{Now defining } p = \frac{2.5-1}{1} = 1.5$$
$$f(2.5, 3.5) = 9.75 + (1.5)(3) + \frac{(1.5)(0.5)}{2}(2) = 15$$

Cont !

From the functional relation, we also find that

$$f(2.5, 3.5) = (2.5)^2 + (3.5)^2 - 3.5 = 15$$

And hence no error in interpolation!!!

## DIFFERENTIATION USING DIFFERENCE OPERATORS

### Introduction

Consider a function of single variable  $y = f(x)$ . If the function is known and simple, we can easily obtain its derivative (s) or can evaluate its definite integral

However, if we do not know the function as such or the function is complicated and is given in a tabular form at a set of points  $x_0, x_1, \dots, x_n$ , we use only numerical methods for differentiation or integration of the given function.

We shall discuss numerical approximation to derivatives of functions of two or more variables in the lectures to follow when we shall talk about partial differential equations. In what follows, we shall derive and illustrate various formulae for numerical differentiation of a function of a single variable based on finite difference operators and interpolation.

Subsequently, we shall develop Newton-Cotes formulae and related trapezoidal rule and Simpson's rule for numerical integration of a function.

We assume that the function  $y = f(x)$  is given for the values of the independent variable  $x = x_0 + ph$ , for  $p = 0, 1, 2, \dots$  and so on. To find the derivatives of such a tabular function, we proceed as follows

#### Case I:

Using forward difference operator  $\Delta$  and combining equations

$$E f(x) = f(x+h)$$

and

$$\Delta = E - 1$$

$$hD = \log E = \log(1 + \Delta)$$

Remember the Differential operator,  $D$  is known to represents the property

$$\left. \begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned} \right\}$$

This would mean that in terms of  $\Delta$  :

$$D = \frac{1}{h} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \dots \right)$$

Therefore

$$Df(x_0) = f'(x_0) = \frac{1}{h} \left[ \Delta f(x_0) - \frac{\Delta^2 f(x_0)}{2} + \frac{\Delta^3 f(x_0)}{3} - \frac{\Delta^4 f(x_0)}{4} + \frac{\Delta^5 f(x_0)}{5} - \dots \right] = \frac{d}{dx} f(x_0)$$

in other words

$$Dy_0 = y'_0 = \frac{1}{h} \left( \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots \right)$$

Also, we can easily verify

$$\begin{aligned} D^2 &= \frac{1}{h^2} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right)^2 \\ &= \frac{1}{h^2} \left( \Delta^2 - \Delta^3 + \frac{11}{12} \Delta^4 - \frac{5}{6} \Delta^5 + \dots \right) \\ D^2 y_0 &= \frac{d^2 y}{dx^2} = y''_0 = \frac{1}{h^2} \left( \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + \dots \right) \end{aligned}$$

### Case II:

Using backward difference operator, we have  $\nabla$

$$hD = -\log(1 - \nabla)$$

On expansion, we have

$$D = \frac{1}{h} \left( \nabla - \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \dots \right)$$

we can also verify that

$$\begin{aligned} D^2 &= \frac{1}{h^2} \left( \nabla - \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \frac{\nabla^4}{4} + \dots \right)^2 \\ &= \frac{1}{h^2} \left( \nabla^2 - \nabla^3 + \frac{11}{12} \nabla^4 - \frac{5}{6} \nabla^5 + \dots \right) \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dx} y_n &= Dy_n = y'_n \\ &= \frac{1}{h} \left( \nabla y_n - \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} - \frac{\nabla^4 y_n}{4} + \dots \right) \end{aligned}$$

and

$$\begin{aligned} y''_n &= D^2 y_n \\ &= \frac{1}{h^2} \left( \nabla^2 y_n - \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n - \frac{5}{6} \nabla^5 y_n + \dots \right) \end{aligned}$$

The formulae for  $Dy_0$  and  $D^2 y_0$  are useful to calculate the first and second derivatives at the beginning of the table of values in terms of forward differences; while formulae for  $y'_n$  and  $y''_n$

The formulae for  $Dy_0$  and  $D^2y_0$  are useful to calculate the first and second derivatives at the beginning of the table of values in terms of forward differences; while formulae for  $y'_n$  and  $y''_n$

**Case III:** Using central difference operator  $\delta$  and following the definitions of differential operator  $D$ , central difference operator  $\delta$  and the shift operator  $E$ , we have

$$\begin{aligned}\delta &= E^{1/2} - E^{-1/2} = e^{hD/2} - e^{-hD/2} \\ &= 2 \sinh \frac{hD}{2}\end{aligned}$$

Therefore, we have

$$\frac{hD}{2} = \sinh^{-1} \frac{\delta}{2}$$

But,

$$\begin{aligned}\sinh^{-1} x &= x - \frac{1}{2} x^3 + \frac{1 \times 3}{2 \times 4} x^5 \\ &\quad - \frac{1 \times 3 \times 5}{2 \times 4 \times 6} x^7 + \dots\end{aligned}$$

Using the last two equations we get

$$\frac{hD}{2} = \left( \frac{\delta}{2} - \frac{\delta^3}{6 \times 8} + \frac{3}{40 \times 32} \delta^5 - \dots \right)$$

That is,

$$D = \frac{1}{h} \left( \delta - \frac{1}{24} \delta^3 + \frac{3}{640} \delta^5 - \dots \right)$$

Therefore

$$\begin{aligned}\frac{d}{dx} y &= y' = Dy \\ &= \frac{1}{h} \left( \delta y - \frac{1}{24} \delta^3 y + \frac{3}{640} \delta^5 y - \dots \right)\end{aligned}$$

Also

$$D^2 = \frac{1}{h^2} \left( \delta^2 - \frac{1}{12} \delta^4 + \frac{3}{90} \delta^6 - \dots \right)$$

Hence

$$y'' = D^2 y = \frac{1}{h^2} \left( \delta^2 y - \frac{1}{12} \delta^4 y + \frac{1}{90} \delta^6 y - \dots \right)$$

For calculating the second derivative at an interior tabular point, we use the equation

$$D^2 = \frac{1}{h^2} \left( \delta^2 - \frac{1}{12} \delta^4 + \frac{3}{90} \delta^6 - \dots \right)$$

while for computing the first derivative at an interior tabular point, we in general use another convenient form for D, which is derived as follows. Multiply the right hand side of

$$\frac{d}{dx} y = y' = Dy = \frac{1}{h} \left( \delta y - \frac{1}{24} \delta^3 y + \frac{3}{640} \delta^5 y - \dots \right)$$

by

$$\frac{\mu}{\sqrt{1 + (\delta^2/4)}}$$

which is unity and noting the Binomial expansion

$$\left( 1 + \frac{1}{4} \delta^2 \right)^{-1/2} = 1 - \frac{1}{8} \delta^2 + \frac{3}{128} \delta^4 - \frac{15}{48 \times 64} \delta^6 + \dots$$

we get

$$D = \frac{\mu}{h} \left( 1 - \frac{1}{8} \delta^2 + \frac{3}{128} \delta^4 - \dots \right) \left( \delta y - \frac{1}{24} \delta^3 y + \frac{3}{640} \delta^5 y - \dots \right)$$

On simplification, we obtain

$$D = \frac{\mu}{h} \left( \delta y - \frac{1}{6} \delta^3 y + \frac{4}{120} \delta^5 y - \dots \right)$$

Therefore the equation can also be written in another useful form as

$$y' = D = \frac{\mu}{h} \left( \delta y - \frac{1}{6} \delta^3 y + \frac{1}{30} \delta^5 y - \dots \right)$$

The last two equations for  $y''$  and  $y'$  respectively are known as Stirling's formulae for computing the derivatives of a tabular function. Similar formulae can be derived for computing higher order derivatives of a tabular function.

The equation for  $y'$  can also be

written as

$$y' = \frac{\mu}{h} \left[ \delta y - \frac{1^2}{3!} \delta^3 y + \frac{(1)^2(2)^2}{5!} \delta^5 y - \frac{(1)^2(2)^2(3)^2}{7!} \delta^7 y + \dots \right]$$

In order to illustrate the use of formulae derived so far, for computing the derivatives of a tabulated function, we shall consider some examples in the next lecture.



## DIFFERENTIATION USING DIFFERENCE OPERATORS:

### Applications:

Remember Using forward difference operator  $\Delta$ , the shift operator, the backward difference operator and the average difference operator, we obtained the following formulae:

$$\begin{aligned}
 Dy_0 = y'_0 &= \frac{1}{h} \left( \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots \right) \\
 D^2 y_0 = \frac{d^2 y_0}{dx^2} = y''_0 &= \frac{1}{h^2} \left( \Delta^2 y_0 - \frac{11}{12} \Delta^3 y_0 + \frac{5}{6} \Delta^4 y_0 - \dots \right) \\
 \frac{d}{dx} y_n = Dy_n = y'_n &= \frac{1}{h} \left( \nabla y_n - \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} - \frac{\nabla^4 y_n}{4} + \dots \right) \\
 y''_n = D^2 y_n &= \frac{1}{h^2} \left( \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n + \dots \right) \\
 y'' = D^2 y &= \frac{1}{h^2} \left( \delta^2 y - \frac{1}{12} \delta^4 y + \frac{1}{90} \delta^6 y - \dots \right) \\
 \frac{d}{dx} y = y' = Dy &= \frac{1}{h} \left( \delta y - \frac{1}{24} \delta^3 y + \frac{3}{640} \delta^5 y - \dots \right)
 \end{aligned}$$

Recall from what we mentioned in the last lecture that for calculating the second derivative at an interior tabular point, we use the equation

$$D^2 = \frac{1}{h^2} \left( \delta^2 - \frac{1}{12} \delta^4 + \frac{3}{90} \delta^6 - \dots \right)$$

While for computing the first derivative at an interior tabular point, we in general use another convenient form for  $D$ , which is derived as follows. Multiply the right hand side of

$$\frac{d}{dx} y = y' = Dy = \frac{1}{h} \left( \delta y - \frac{1}{24} \delta^3 y + \frac{3}{640} \delta^5 y - \dots \right)$$

By

$$\frac{\mu}{\sqrt{1 + (\delta^2/4)}}$$

Which is unity and noting the Binomial expansion

$$\left( 1 + \frac{\delta^2}{4} \right)^{-1/2} = 1 - \frac{1}{8} \delta^2 + \frac{3}{128} \delta^4 - \frac{15}{48 \times 64} \delta^6 + \dots$$

We get

$$D = \frac{\mu}{h} \left( 1 - \frac{1}{8} \delta^2 + \frac{3}{128} \delta^4 - \dots \right) \left( \delta - \frac{1}{24} \delta^3 + \frac{3}{640} \delta^5 - \dots \right)$$

Simplification we get

$$D = \frac{\mu}{h} \left( \delta - \frac{1}{6} \delta^3 + \frac{4}{120} \delta^5 - \dots \right)$$

Therefore the equation can also be written in another useful form as

$$y' = D = \frac{\mu}{h} \left( \delta y - \frac{1}{6} \delta^3 y + \frac{1}{30} \delta^5 y - \dots \right)$$

The last two equations  $y''$  and  $y'$  respectively are known as Stirling's formulae for computing the derivatives of a tabular function. Similar formulae can be derived for computing higher order derivatives of a tabular function.

The equation for  $y'$  can also be written as

$$y' = \frac{\mu}{h} \left[ \delta y - \frac{1^2}{3!} \delta^3 y + \frac{(1)^2(2)^2}{5!} \delta^5 y - \frac{(1)^2(2)^2(3)^2}{7!} \delta^7 y + \dots \right]$$

In order to illustrate the use of formulae derived so far, for computing the derivatives of a tabulated function, we consider the following example :

**Example:**

Compute  $f''(0)$  and  $f'(0.2)$  from the following tabular data.

|      |      |      |      |       |       |        |
|------|------|------|------|-------|-------|--------|
| x    | 0.0  | 0.2  | 0.4  | 0.6   | 0.8   | 1.0    |
| F(x) | 1.00 | 1.16 | 3.56 | 13.96 | 41.96 | 101.00 |

### Solution

Since  $x = 0$  and  $0.2$  appear at and near beginning of the table, it is appropriate to use formulae based on forward differences to find the derivatives. The difference table for the given data is depicted below:

| x   | f(x)   | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ | $\Delta^5 f(x)$ |
|-----|--------|---------------|-----------------|-----------------|-----------------|-----------------|
| 0.0 | 1.00   |               |                 |                 |                 |                 |
| 0.2 | 1.16   | 0.16          |                 |                 |                 |                 |
| 0.4 | 3.56   | 2.40          | 2.24            | 5.76            |                 |                 |
| 0.6 | 13.96  | 10.40         | 8.00            | 9.60            | 3.84            | 0.0             |
| 0.8 | 41.96  | 28.00         | 17.60           | 13.44           | 3.84            |                 |
| 1.0 | 101.00 | 59.04         | 31.04           |                 |                 |                 |

Using forward difference formula for  $D^2 f(x)$ ,

**i.e**

$$D^2 f(x) = \frac{1}{h^2} \left[ \Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12} \Delta^4 f(x) - \frac{5}{6} \Delta^5 f(x) \right]$$

**we obtained**

$$f''(0) = \frac{1}{(0.2)^2} \left[ 2.24 - 5.76 + \frac{11}{12} (3.84) - \frac{5}{6} (0) \right] = 0.0$$

Also, using the formula for  $Df(x)$ , we have

$$Df(x) = \frac{1}{h} \left[ \Delta f(x) - \frac{\Delta^2 f(x)}{2} + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4} \right]$$

Hence,

$$f'(0.2) = \frac{1}{0.2} \left( 2.40 - \frac{8.00}{2} + \frac{9.60}{3} - \frac{3.84}{4} \right) = 3.2$$

### Example

Find  $y'(2.2)$  and  $y''(2.2)$  from the table.

|      |        |        |        |        |        |
|------|--------|--------|--------|--------|--------|
| x    | 1.4    | 1.6    | 1.8    | 2.0    | 2.2    |
| Y(x) | 4.0552 | 4.9530 | 6.0496 | 7.3891 | 9.0250 |

**Solution:**

Since  $x=2.2$  occurs at the end of the table, it is appropriate to use backward difference formulae for derivatives. The backward difference table for the given data is shown below:

| x   | f(x)   | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|-----|--------|------------|--------------|--------------|--------------|
| 1.4 | 4.0552 |            |              |              |              |
| 1.6 | 4.9530 | 0.8978     |              |              |              |
| 1.8 | 6.0496 | 1.0966     | 0.1988       | 0.0441       |              |
| 2.0 | 7.3891 | 1.3395     | 0.2429       | 0.0535       | 0.0094       |
| 2.2 | 9.0250 | 1.6359     | 0.2964       |              |              |

Using backward difference formulae for  $y'(x)$  and  $y''(x)$ , we have

$$y'_n = \frac{1}{h} \left( \nabla y_n + \frac{\nabla^2 y}{2} + \frac{\nabla^3 y}{3} + \frac{\nabla^4 y}{4} \right)$$

Therefore,

$$y'(2.2) = \frac{1}{0.2} \left( 1.6359 + \frac{0.2964}{2} + \frac{0.0535}{3} + \frac{0.0094}{4} \right) = 5(1.8043) = 9.02$$

Also

$$y_n'' = \frac{1}{h^2} \left( \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n \right)$$

Therefore

$$y''(2.2) = \frac{1}{(0.2)^2} \left( 0.2964 + 0.0535 + \frac{11}{12} (0.0094) \right)$$

$$= 25(0.3585) = 8.9629$$

### Example

Given the table of values, estimate,  $y''(1.3)$

|   |        |        |        |        |        |        |
|---|--------|--------|--------|--------|--------|--------|
| x | 1.3    | 1.5    | 1.7    | 1.9    | 2.1    | 2.3    |
| y | 2.9648 | 2.6599 | 2.3333 | 1.9922 | 1.6442 | 1.2969 |

**Solution** Since  $x = 1.3$  appear at beginning of the table, it is appropriate to use formulae based on forward differences to find the derivatives. The difference table for the given data is depicted below:

| x   | y      | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ | $\Delta^5 f(x)$ |
|-----|--------|---------------|-----------------|-----------------|-----------------|-----------------|
| 1.3 | 2.9648 |               |                 |                 |                 |                 |
| 1.5 | 2.6599 | -0.3049       |                 |                 |                 |                 |
| 1.7 | 2.3333 | -0.3266       | -0.0217         |                 |                 |                 |
| 1.9 | 1.9922 | -0.3411       | -0.0145         | 0.0072          |                 |                 |
| 2.1 | 1.6442 | -0.348        | -0.0069         | 0.0076          | 0.0004          |                 |
| 2.3 | 1.2969 | -0.3473       | 0.0007          | 0.0076          | 0               | -0.0004         |

$H=0.2$

Using forward difference formula for  $D^2 f(x)$ ,

$$D^2 f(x) = \frac{1}{h^2} \left[ \Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12} \Delta^4 f(x) - \frac{5}{6} \Delta^5 f(x) \right]$$

We obtain

$$f''(1.3) = \frac{1}{(0.2)^2} \left[ -0.0217 - 0.0072 + \frac{11}{12}(0.0004) - \frac{5}{6}(-0.0004) \right]$$

$$f''(1.3) = \frac{1}{(0.2)^2} [-0.0217 - 0.0072 + (0.9167)(0.0004) - (0.8334)(-0.0004)]$$

$$f''(1.3) = \frac{1}{(0.2)^2} [-0.0217 - 0.0072 + (0.0003667) + (0.0003334)]$$

$$f''(1.3) = \frac{1}{(0.2)^2} [-0.0282] = -0.7050$$

**Case IV:** Derivation of Two and three point formulae:  
Retaining only the first term in equation:

$$Dy_0 = y'_0 = \frac{1}{h} \left( \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots \right)$$

we can get another useful form for the first derivative as

$$y'_i = \frac{\Delta y_i}{h} = \frac{y_{i+1} - y_i}{h} = \frac{y(x_i + h) - y(x_i)}{h}$$

Similarly, by retaining only the first term in Eqn.

$$\frac{d}{dx} y_n = Dy_n = y'_n = \frac{1}{h} \left( \nabla y_n - \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} - \frac{\nabla^4 y_n}{4} + \dots \right)$$

$$y'_i = \frac{\nabla y_i}{h} = \frac{y_i - y_{i-1}}{h} = \frac{y(x_i) - y(x_i - h)}{h}$$

Adding the last two equations, we have

$$y'_i = \frac{y(x_i + h) - y(x_i - h)}{2h}$$

These Equations constitute two-point formulae for the first derivative. By retaining only the first term in Equation ,

$$D^2 y_0 = \frac{d^2 y}{dx^2} = y''_0 = \frac{1}{h^2} \left( \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 + \frac{5}{6} \Delta^5 y_0 + \dots \right)$$

we get,

$$y''_i = \frac{\Delta^2 y_i}{h^2} = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} = \frac{y(x_i + 2h) - 2y(x_i + h) + y(x_i)}{h^2}$$

Similarly we get

$$y''_i = \frac{\Delta^2 y_i}{h^2} = \frac{y(x_i) - 2y(x_i - h) + y(x_i - 2h)}{h^2}$$

While retaining only the first term in the expression for  $y''$  in terms of  $\delta$  we obtain

$$\begin{aligned}
 y_i'' = \frac{\delta^2 y_i}{h^2} &= \frac{\delta y_{i+(1/2)} - \delta y_{i-(1/2)}}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \\
 &= \frac{y(x_i + h) - 2y(x_i) + y(x_i - h)}{h^2}
 \end{aligned}$$

The last three equations constitute three-point formulae for computing the second derivative. We shall see later that these two- and three-point formulae become handy for developing extrapolation methods to numerical differentiation and integration.

**Example 1;**

Write first derivative of  $f(x)$  at  $x = 0.1, 0.2, 0.3$   
where  $f(x)$  is given by

|      |       |       |       |       |       |       |
|------|-------|-------|-------|-------|-------|-------|
| X    | 0.1   | 0.2   | 0.3   | 0.4   | 0.5   | 0.6   |
| F(x) | 0.425 | 0.475 | 0.400 | 0.452 | 0.525 | 0.575 |

**solution;**

Using the two point equation we have

$$f'(0.1) = \frac{f(0.2) - f(0.1)}{h} = \frac{0.475 - 0.425}{0.1} = \frac{0.050}{0.1} = 0.5.$$

$$f'(0.2) = \frac{f(0.3) - f(0.2)}{h} = \frac{0.400 - 0.475}{0.1} = \frac{-0.075}{0.1} = -0.75.$$

$$f'(0.3) = \frac{f(0.4) - f(0.3)}{h} = \frac{0.450 - 0.400}{0.1} = \frac{0.05}{0.1} = 0.5.$$

**Example 2:**

Find the 2<sup>nd</sup> derivative at 0.3, 0.4, 0.5 for the function given in the example above.

**Solution;**

$$f''(0.3) = \frac{f(0.4) - 2f(0.3) + f(0.2)}{h^2} = \frac{0.125}{0.01} = 12.5$$

$$f''(0.4) = \frac{f(0.5) - 2f(0.4) + f(0.3)}{h^2} = \frac{0.025}{0.01} = 2.5$$

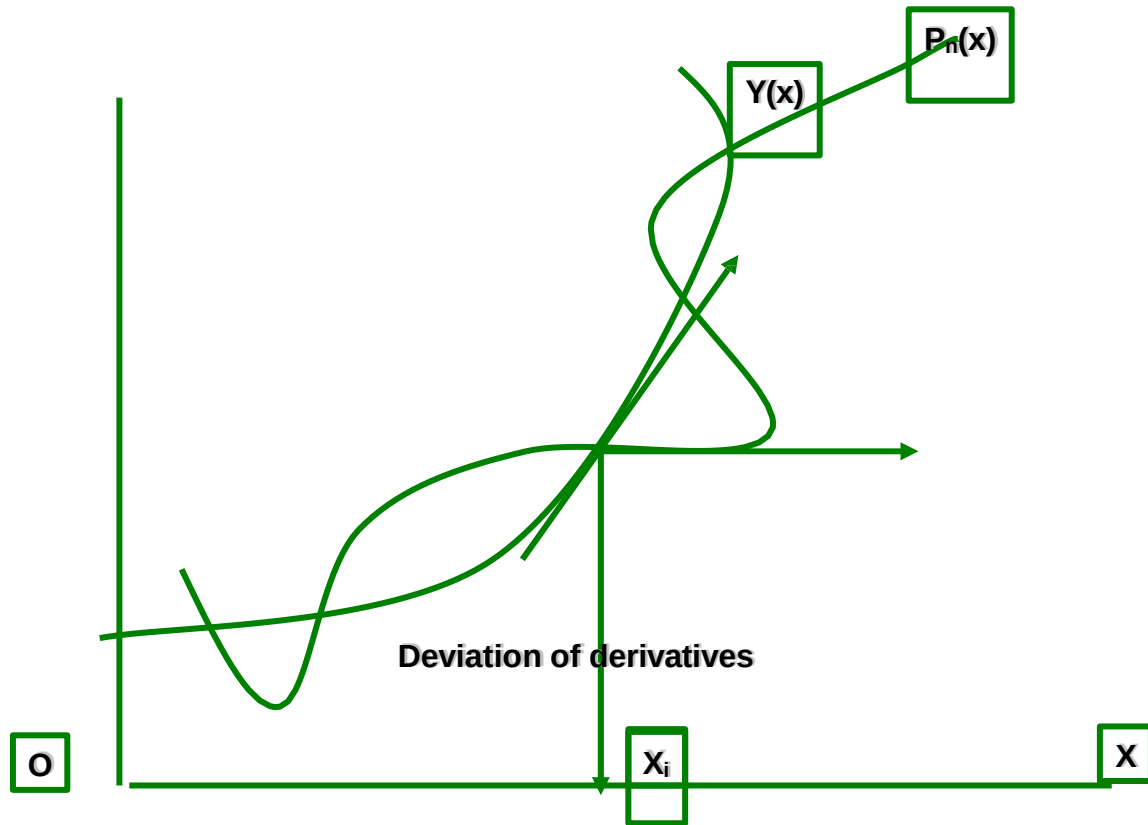
$$f''(0.5) = \frac{f(0.6) - 2f(0.5) + f(0.4)}{h^2} = \frac{0.075}{0.01} = 7.5$$

**DIFFERENTIATION USING INTERPOLATION**

If the given tabular function  $y(x)$  is reasonably well approximated by a polynomial  $P_n(x)$

of degree  $n$ , it is hoped that the result of  $P_n'(x)$  will also satisfactorily approximate the corresponding derivative of  $y(x)$ .

However, even if  $P_n(x)$  and  $y(x)$  coincide at the tabular points, their derivatives or slopes may substantially differ at these points as is illustrated in the Figure below:



For higher order derivatives, the deviations may be still worst. However, we can estimate the error involved in such an approximation.

For non-equidistant tabular pairs  $(x_i, y_i)$ ,  $i = 0, \dots, n$  we can fit the data by using either

Lagrange's interpolating polynomial or by using Newton's divided difference interpolating polynomial. In view of economy of computation, we prefer the use of the latter polynomial.

Thus, recalling the Newton's divided difference interpolating polynomial for fitting this data as

$$P_n(x) = y[x_0] + (x - x_0)y[x_0, x_1] + (x - x_0)(x - x_1)y[x_0, x_1, x_2] \\ + \dots + \prod_{i=0}^{n-1} (x - x_i)y[x_0, x_1, \dots, x_n]$$

Assuming that  $P_n(x)$  is a good approximation to  $y(x)$ , the polynomial approximation to  $y'(x)$  can be obtained by differentiating  $P_n(x)$ . Using product rule of differentiation, the derivative of the products in  $P_n(x)$  can be seen as follows:

$$\frac{d}{dx} \prod_{i=0}^{n-1} (x - x_i) = \sum_{i=0}^{n-1} \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{x - x_i}$$

**Thus,  $y'(x)$  is approximated by  $P'_n(x)$  which is given by**

$$P'_n(x) = y[x_0, x_1] + [(x - x_1) + (x - x_0)]y[x_0, x_1, x_2] + \cdots + \sum_{i=0}^n \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{x - x_i} y[x_0, x_1, \dots, x_n]$$

The error estimate in this approximation can be seen from the following.

We have seen that if  $y(x)$  is approximated by  $P_n(x)$ , the error estimate is shown to be

$$E_n(x) = y(x) - P_n(x) = \frac{\prod(x)}{(n+1)!} y^{(n+1)}(\xi)$$

Its derivative with respect to  $x$  can be written as

$$E'_n(x) = y'(x) - P'_n(x) = \frac{\prod'(x)}{(n+1)!} y^{(n+1)}(\xi) + \frac{\prod(x)}{(n+1)!} \frac{d}{dx} y^{(n+1)}(\xi)$$

Its derivative with respect to  $x$  can be written as

$$E'_n(x) = y'(x) - P'_n(x) = \frac{\prod'(x)}{(n+1)!} y^{(n+1)}(\xi) + \frac{\prod(x)}{(n+1)!} \frac{d}{dx} y^{(n+1)}(\xi)$$

Since  $\xi(x)$  depends on  $x$  in an unknown way the derivative

$$\frac{d}{dx} y^{(n+1)}(\xi)$$

cannot be evaluated. However, for any of the tabular points  $x = x_i$ ,  $\prod(x)$  vanishes and the difficult term drops out. Thus, the error term in the last equation at the tabular point  $x = x_i$  simplifies to

$$E'_{n,i}(x) = \text{Error} = \frac{\prod'(x_i)}{(n+1)!} y^{(n+1)}(\xi)$$

for some  $\xi$  in the interval  $I$  defined by the smallest and largest of  $x_0, x_1, \dots, x_n$  and

$$\prod'(x_i) = (x_i - x_0) \cdots (x_i - x_n) = \prod_{\substack{j=0 \\ j \neq i}}^n (x_i - x_j)$$

The error in the  $r$ -th derivative at the tabular points can indeed be expressed analogously.

To understand this method better, we consider the following example.

**Example**

Find  $f'(0.25)$  and  $f'(0.22)$  from the following data using the method based on divided differences:

|        |        |        |        |
|--------|--------|--------|--------|
| X      | 0.15   | 0.21   | 0.23   |
| Y=f(x) | 0.1761 | 0.3222 | 0.3617 |

|        |        |        |        |
|--------|--------|--------|--------|
| X      | 0.27   | 0.32   | 0.35   |
| Y=f(x) | 0.4314 | 0.5051 | 0.5441 |

**Solution** We first construct divided difference table for the given data as shown below:

| X    | Y      | 1 <sup>st</sup> D.D | 2 <sup>nd</sup> D.D | 3 <sup>rd</sup> D.D |
|------|--------|---------------------|---------------------|---------------------|
| 0.15 | 0.1761 |                     |                     |                     |
| 0.21 | 0.3222 | 2.4350              | -5.700              |                     |
| 0.23 | 0.3617 | 1.9750              | -3.8750             | 15.6250             |
| 0.27 | 0.4314 | 1.7425              | -2.9833             | 8.1064              |
| 0.32 | 0.5051 | 1.4740              | -2.1750             | 6.7358              |
| 0.35 | 0.5441 | 1.3000              |                     |                     |

Using divided difference formula

$$\sum_{i=0}^n \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{x-x_i} y[x_0, x_1, \dots, x_n]$$

from a quadratic polynomial, we have

$$y'(x) = P_3'(x) = y[x_0, x_1] + \{(x-x_1) + (x-x_0)\}y[x_0, x_1, x_2] \\ + \{(x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)\}y[x_0, x_1, x_2, x_3]$$

Thus, using first, second and third differences from the table, the above equation yields

$$y'(0.25) = 2.4350 + [(0.25-0.21) + (0.25-0.15)](-5.75) \\ + [(0.25-0.21)(0.25-0.23) + (0.25-0.15)(0.25-0.23) \\ + (0.25-0.15)(0.25-0.21)](15.625)$$

Therefore

$$f'(0.25) = 2.4350 - 0.805 + 0.10625 \\ = 1.7363$$

Similarly we can show that

$$f'(0.22) = 1.9734$$

**Example**

The following divided difference table is for  $y=1/x$ . use it to find  $y'(0.75)$  from a quadratic polynomial Fit.

| x    | Y=1/x  | 1 <sup>st</sup> divided difference | 2 <sup>nd</sup> divided difference | 3 <sup>rd</sup> divided difference |
|------|--------|------------------------------------|------------------------------------|------------------------------------|
| .25  | 4.0000 | -8.0000                            |                                    |                                    |
| .50  | 2.0000 | -2.6668                            | 10.6664                            |                                    |
| .75  | 1.3333 | -1.3332                            | 2.6672                             | -14.2219                           |
| 1.00 | 1.0000 | -.8000                             | 1.0664                             | -2.1344                            |
| 1.25 | .8000  | -.5332                             | 0.5336                             | -0.7104                            |
| 1.50 | .6667  |                                    |                                    |                                    |

**Solution:**

Using divided difference formula

$$\sum_{i=0}^n \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{x-x_i} y[x_0, x_1, \dots, x_n]$$

If we find the values for quadratic fit then we use the following formula,

$$y'(x) = P_3'(x)$$

$$= y[x_0, x_1] + \{(x-x_1) + (x-x_0)\} y[x_0, x_1, x_2] \\ + \{(x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)\} y[x_0, x_1, x_2, x_3]$$

Thus, using first, second, third differences from the table, we get

$$y'(x) = P_3'(x) = -8.0000 + \{(x-.50) + (x-.25)\} 10.6664 \\ + \{(x-.50)(x-.75) + (x-.25)(x-.75) + (x-.25)(x-.50)\} (-14.2219)$$

Put  $x = 0.75$

$$y'(0.75) = P_3'(0.75) = -8.0000 + \{(0.75-.50) + (0.75-.25)\} 10.6664 \\ + \{(0.75-.50)(0.75-.75) + (0.75-.25)(0.75-.75) + (0.75-.25)(0.75-.50)\} (-14.2219)$$

$$y'(0.75) = P_3'(0.75) = -8.0000 + \{(0.25) + (0.50)\} 10.6664 \\ + \{(0.25)(0) + (0.50)(0) + (0.50)(0.25)\} (-14.2219)$$

$$y'(0.75) = P_3'(0.75) = -8.0000 + \{0.75\} 10.6664 + \{0.125\} (-14.2219)$$

$$y'(0.75) = P_3'(0.75) = -8.0000 + 7.9998 - 1.7777375$$

$$y'(0.75) = P_3'(0.75) = -1.7779$$



### RICHARDSON'S EXTRAPOLATION METHOD

To improve the accuracy of the derivative of a function, which is computed by starting with an arbitrarily selected value of  $h$ , Richardson's extrapolation method is often employed in practice, in the following manner:

Suppose we use two-point formula to compute the derivative of a function, then we have

$$\begin{aligned} y'(x) &= \frac{y(x+h) - y(x-h)}{2h} + E_T \\ &= F(h) + E_T \end{aligned}$$

Where  $E_T$  is the truncation error. Using Taylor's series expansion, we can see that

$$E_T = c_1 h^2 + c_2 h^4 + c_3 h^6 + \dots$$

The idea of Richardson's extrapolation is to combine two computed values of  $y'(x)$  using the same method but with two different step sizes usually  $h$  and  $h/2$  to yield a higher order method. Thus, we have

$$y'(x) = F(h) + c_1 h^2 + c_2 h^4 + \dots$$

And

$$y'(x) = F\left(\frac{h}{2}\right) + c_1 \frac{h^2}{4} + c_2 \frac{h^4}{16} + \dots$$

Here,  $c_i$  are constants, independent of  $h$ , and  $F(h)$  and  $F(h/2)$  represent approximate values of derivatives. Eliminating  $c_1$  from the above pair of equations, we get

$$y'(x) = \frac{4F\left(\frac{h}{2}\right) - F(h)}{3} + d_1 h^4 + O(h^6)$$

Now assuming that

$$F\left(\frac{h}{2}\right) = \frac{4F\left(\frac{h}{2}\right) - F(h)}{3}$$

Equation for  $y'(x)$  above reduces to

$$y'(x) = F\left(\frac{h}{2}\right) + d_1 h^4 + O(h^6)$$

Thus, we have obtained a fourth-order accurate differentiation formula by combining two results which are of second-order accurate. Now, repeating the above argument, we have

$$\begin{aligned} y'(x) &= F\left(\frac{h}{2}\right) + d_1 h^4 + O(h^6) \\ y'(x) &= F_1\left(\frac{h}{4}\right) + \frac{d_1 h^4}{16} + O(h^6) \end{aligned}$$

Eliminating  $d_1$  from the above pair of equations, we get a better approximation as

$$y'(x) = F_2\left(\frac{h}{4}\right) + O(h^6)$$

Which is of sixth-order accurate, where

$$F_2\left(\frac{h}{4}\right) = \frac{4^2 F_1\left(\frac{h}{2}\right) - F_1\left(\frac{h}{2}\right)}{4^2 - 1}$$

This extrapolation process can be repeated further until the required accuracy is achieved, which is called an extrapolation to the limit. Therefore the equation for  $F_2$  above can be generalized as

$$F_m\left(\frac{h}{2^m}\right) = \frac{4^m F_{m-1}\left(\frac{h}{2^m}\right) - F_{m-1}\left(\frac{h}{2^{m-1}}\right)}{4^m - 1},$$

$m = 1, 2, 3, \dots$

Where  $F_0(h) = F(h)$ .

To illustrate this procedure, we consider the following example.

**Example:** Using the Richardson's extrapolation limit, find  $y'(0.05)$  to the function  $y = -1/x$ , with  $h = 0.0128, 0.0064, 0.0032$ .

**Solution**

To start with, we take,  $h = 0.0128$ , then compute  $F(h)$  as

$$\begin{aligned} F(h) &= \frac{y(x+h) - y(x-h)}{2h} = \frac{-\frac{1}{0.05+0.0128} + \frac{1}{0.05-0.0128}}{2(0.0128)} \\ &= \frac{-15.923566 + 26.88172}{0.0256} \\ &= 428.05289 \end{aligned}$$

Similarly,  $F(h/2) = 406.66273$ . Therefore, using Eq. (7.30), we get

$$F_1\left(\frac{h}{2}\right) = \frac{4F\left(\frac{h}{2}\right) - F\left(\frac{h}{2}\right)}{4 - 1} = 399.5327$$

Which is accurate to  $O(h^4)$ . Halving the step size further, we compute

$$\begin{aligned} F\left(\frac{h}{2^2}\right) &= \frac{-\frac{1}{0.05+0.0032} + \frac{1}{0.05-0.0032}}{2(0.0032)} \\ &= 401.64515 \end{aligned}$$

And

$$F_1\left(\frac{h}{2^2}\right) = \frac{4F\left(\frac{h}{2}\right) - F\left(\frac{h}{2}\right)}{4-1}$$

$$= 399.97263$$

Again, using Eq. , we obtain

$$F_2\left(\frac{h}{2^2}\right) = \frac{4^2 F_1\left(\frac{h}{2^2}\right) - F_1\left(\frac{h}{2}\right)}{4^2 - 1}$$

$$= 400.00195$$

The above computation can be summarized in the following table:

| h       | F        | F1       | F2        |
|---------|----------|----------|-----------|
| 0.0128  | 428.0529 |          |           |
| 0.00064 | 406.6627 | 399.5327 | 400.00195 |
| 0.0032  | 401.6452 | 399.9726 |           |

Thus, after two steps, it is found that  $y'(0.05) = 400.00915$  while the exact value is

$$y'(0.05) = \left(\frac{1}{x^2}\right)_{x=0.05} = \frac{1}{0.0025} = 400$$

## Numerical Differentiation and Integration

### INTRODUCTION

DIFFERENTIATION USING DIFFERENCE OPERATORS

DIFFERENTIATION USING INTERPOLATION

RICHARDSON'S EXTRAPOLATION METHOD

NUMERICAL INTEGRATION

NEWTON-COTES INTEGRATION FORMULAE

THE TRAPEZOIDAL RULE ( COMPOSITE FORM)

SIMPSON'S RULES (COMPOSITE FORM)

ROMBERG'S INTEGRATION

DOUBLE INTEGRATION

Basic Issues in Integration What does an integral represent?

$$AREA = \int_a^b f(x) dx$$

$$VOLUME = \int_c^d \int_a^b g(x, y) dx dy$$

Basic definition of an integral::

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x$$

Sum of Height x Width

### Objective:

Evaluate  $I = \int_a^b f(x) dx$  without doing calculation analytically.

When would we want to do this?

1. Integrand is too complicated to integrate analytically.

$$\int_0^2 \frac{2 + \cos(1 + \sqrt{x})}{\sqrt{1 + 0.5x}} e^{0.5x} dx$$

2. Integrand is not precisely defined by an equation, i.e., we are given a set of data  $(x_i, f(x_i))$ ,  $i=1, \dots, n$ .

All methods are applicable to integrands that are functions.

Some are applicable to tabulated values.

### Key concepts:

1. Integration is a summing process. Thus virtually all numerical approximations can be represented by

$$I = \int_a^b f(x)dx = \sum_{i=0}^n \Delta x f(x_i) + E_t$$

Where:

$$\begin{aligned} \Delta x &= \text{weights} \\ x_i &= \text{sampling points} \\ E_t &= \text{truncation error} \end{aligned}$$

## 2. Closed & Open forms:

Closed forms include the end points  $a$  &  $b$  in  $x_i$ . Open forms do not.

## NUMERICAL INTEGRATION

Consider the definite integral

$$I = \int_{x=a}^b f(x)dx$$

Where  $f(x)$  is known either explicitly or is given as a table of values corresponding to some values of  $x$ , whether equispaced or not. Integration of such functions can be carried out using numerical techniques.

Of course, we assume that the function to be integrated is smooth and Riemann integrable in the interval of integration. In the following section, we shall develop Newton-Cotes formulae based on interpolation which form the basis for trapezoidal rule and Simpson's rule of numerical integration.

## NEWTON-COTES INTERGRATION FORMULAE

In this method, as in the case of numerical differentiation, we shall approximate the given tabulated function, by a polynomial  $P_n(x)$  and then integrate this polynomial.

Suppose, we are given the data  $(x_i, y_i)$ ,  $i = 0(1)n$ , at equispaced points with spacing  $h = x_{i+1} - x_i$ , we can represent the polynomial by any standard interpolation polynomial.

Suppose, we use Lagrangian approximation, then we have

$$f(x) = \sum L_k(x) y(x_k)$$

With associated error given by

$$E(x) = \frac{\Pi(x)}{(n+1)!} y^{(n+1)}(\xi)$$

Where

$$L_k(x) = \frac{\Pi(x)}{(x - x_k) \Pi'(x_k)}$$

And

$$\Pi(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

Then, we obtain an equivalent integration formula to the definite integral in the form

$$\int_a^b f(x)dx \approx \sum_{k=1}^n c_k y(x_k)$$

Where  $c_k$  are the weighting coefficients given by

$$c_k = \int_a^b L_k(x)dx$$

Which are also called Cotes numbers. Let the equispaced nodes are defined by

$$x_0 = a, x_n = b, h = \frac{b-a}{n}, \text{ and } x_k = x_0 + kh$$

So that  $x_k - x_1 = (k-1)h$  etc. Now, we shall change the variable  $x$  to  $p$  such that,  $x = x_0 + ph$ , then we can rewrite equations.

$$L_k(x) = \frac{\Pi(x)}{(x-x_k)\Pi'(x_k)}$$

$$\Pi(x) = (x-x_0)(x-x_1)\dots(x-x_n)$$

As  $\Pi(x) = h^{n+1} p(p-1)\dots(p-n)$

And 
$$L_k(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$$

$$= \frac{(ph)(p-1)h\dots(p-k+1)h(p-k-1)h\dots(p-n)h}{(kh)(k-1)h\dots(1)h(-1)h\dots(k-n)h}$$

Or

$$L_k(x) = (-1)^{(n-k)} \frac{p(p-1)\dots(p-k+1)(p-k-1)\dots(p-n)}{k!(n-k)!}$$

Also, noting that  $dx = h dp$ . The limits of the integral in Equation

$$c_k = \int_a^b L_k(x)dx$$

change from 0 to  $n$  and equation reduces to

$$c_k = \frac{(-1)^{n-k}(h)}{k!(n-k)!} \int_0^n \dots(p-k+1)(p-k-1)\dots(p-n)dp$$

The error in approximating the integral can be obtained from

$$E_n = \frac{h^{n+2}}{(n+1)!} \int_0^n \dots(p-n)y^{(n+1)}(\xi)dp$$

Where  $x_0 < \xi < x_n$ . For illustration, consider the cases for  $n = 1, 2$ ; For which we get

$$c_0 = -h \int_0^1 (p-1) dp = \frac{h}{2}, c_1 = h \int_0^1 p dp = \frac{h}{2}$$

And

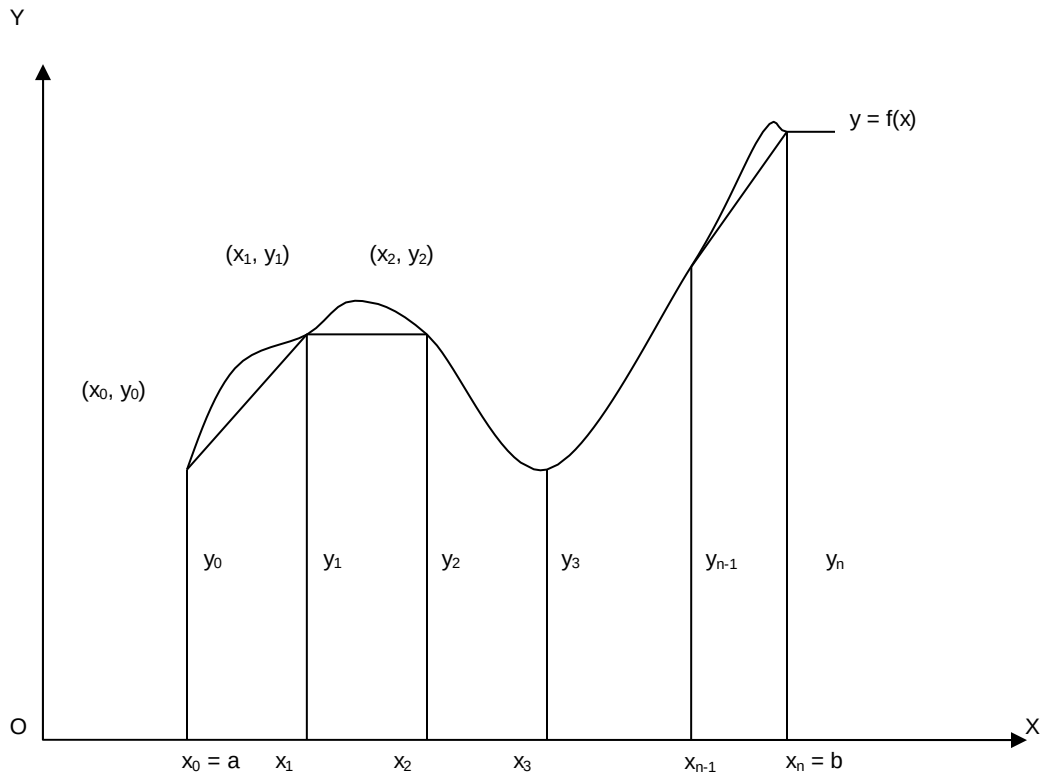
$$E_1 = \frac{h^3}{2} y'(\xi) \int_0^1 p(p-1) dp = -\frac{h^3}{12} y''(\xi)$$

Thus, the integration formula is found to be

$$\int_{x_0}^{x_1} f(x) dx = c_0 y_0 + c_1 y_1 + \text{Error} = \frac{h}{2} (y_0 + y_1) - \frac{h^3}{12} y''(\xi)$$

This equation represents the Trapezoidal rule in the interval  $[x_0, x_1]$  with error term.

Geometrically, it represents an area between the curve  $y = f(x)$ , the x-axis and the ordinates erected at  $x = x_0 (= a)$  and  $x = x_1$  as shown in the figure.



This area is approximated by the trapezium formed by replacing the curve with its secant line drawn between the end points  $(x_0, y_0)$  and  $(x_1, y_1)$ .

For  $n=2$ , We have

$$c_0 = \frac{h^2}{2} \int_0^1 (p-1)(p-2) dp = \frac{h^2}{3}$$

$$c_1 = -h \int_0^1 p(p-2) dp = \frac{4}{3}h$$

$$c_2 = h \int_0^1 p(p-1) dp = \frac{h^2}{3}$$

and the error term is given by

$$E_2 = \frac{h^5}{90} y^{(iv)}(\xi)$$

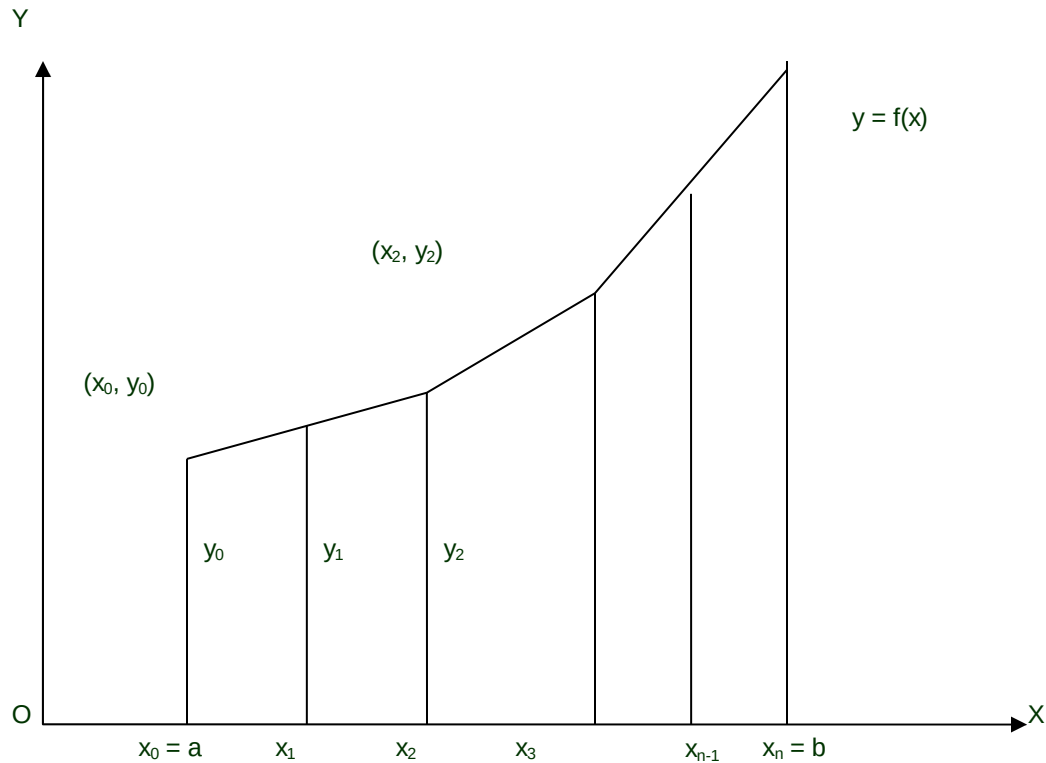
Thus, for  $n = 2$ , the integration takes the form

$$\int_{x_0}^{x_2} f(x) dx = x_0 y_0 + x_1 y_1 + x_2 y_2 + \text{Error}$$

$$= \frac{h}{3} (y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi)$$

This is known as Simpson's 1/3 rule. Geometrically, this equation represents the area between the curve  $y = f(x)$ , the x-axis and the ordinates at  $x = x_0$  and  $x_2$  after replacing the arc of the curve between  $(x_0, y_0)$  and  $(x_2, y_2)$  by an arc of a quadratic polynomial as

in the figure



Thus Simpson's 1/3 rule is based on fitting three points with a quadratic.

Similarly, for  $n = 3$ , the integration is found to be

$$\int_{x_0}^{x_3} f(x) dx = \frac{h}{8} (y_0 + 3y_1 + 3y_2 + y_3) - \frac{3h^5}{80} y^{(iv)}(\xi)$$

This is known as Simpson's 3/8 rule, which is based on fitting four points by a cubic.

Still higher order Newton-Cotes integration formulae can be derived for large values of  $n$ .

But for all practical purposes, Simpson's 1/3 rule is found to be sufficiently accurate.



### The Trapezoidal Rule (Composite Form)

The Newton-Cotes formula is based on approximating  $y = f(x)$  between  $(x_0, y_0)$  and  $(x_1, y_1)$  by a straight line, thus forming a trapezium, is called trapezoidal rule. In order to evaluate the definite integral

$$I = \int_a^b f(x)dx$$

we divide the interval  $[a, b]$  into  $n$  sub-intervals, each of size  $h = (b - a)/n$  and denote the sub-intervals by  $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ , such that  $x_0 = a$  and  $x_n = b$  and  $x_k = x_0 + kh$ ,  $k = 1, 2, \dots, n - 1$ .

Thus, we can write the above definite integral as a sum. Therefore,

$$I = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \dots + \int_{x_{n-1}}^{x_n} f(x)dx$$

The area under the curve in each sub-interval is approximated by a trapezium. The integral  $I$ , which represents an area between the curve  $y = f(x)$ , the  $x$ -axis and the ordinates at  $x = x_0$  and  $x = x_n$  is obtained by adding all the trapezoidal areas in each sub-interval.

Now, using the trapezoidal rule into equation:

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2} (y_0 + y_1) + \frac{h}{2} (y_1 + y_2) + \dots + \frac{h}{2} (y_{n-1} + y_n) + \text{Error}$$

We get

$$I = \int_{x_0}^{x_n} f(x)dx = \frac{h}{2} (y_0 + y_1) - \frac{h^3}{12} y''(\xi_1) + \frac{h}{2} (y_1 + y_2) - \frac{h^3}{12} y''(\xi_2) + \dots + \frac{h}{2} (y_{n-1} + y_n) - \frac{h^3}{12} y''(\xi_n)$$

Where  $x_{k-1} < \xi_k < x_k$ , for  $k = 1, 2, \dots, n - 1$ .

Thus, we arrive at the result

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n) + E$$

Where the error term  $E$  is given by

$$E = -\frac{h^3}{12} [y''(\xi_1) + y''(\xi_2) + \dots + y''(\xi_n)]$$

Equation represents the trapezoidal rule over  $[x_0, x_n]$ , which is also called the composite

form of the trapezoidal rule. The error term given by Equation:

$$E = -\frac{h^3}{12} [y''(\xi_1) + y''(\xi_2) + \dots + y''(\xi_n)]$$

is called the global error.

However, if we assume that  $y''(x)$  is continuous over  $[x_0, x_n]$  then there exists some  $\xi$  in  $[x_0, x_n]$  such that  $x_n = x_0 + nh$  and

$$E_n = -\frac{h^3}{12}[ny'(\xi)] = -\frac{x_n - x_0}{12} h^2 y'(\xi)$$

Then the global error can be conveniently written as  $O(h^2)$ .

### Simpson's Rules (Composite Forms)

In deriving equation. ,

$$\int_{x_0}^{x_2} f(x)dx = x_0y_0 + x_1y_1 + x_2y_2 + \text{Error} = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi) \quad (iv)$$

The Simpson's 1/3 rule, we have used two sub-intervals of equal width. In order to get a composite formula, we shall divide the interval of integration  $[a, b]$  into an even number of sub-intervals say  $2N$ , each of width  $(b-a)/2N$ , thereby we have

$$x_0 = a, x_1, \dots, x_{2N} = b \quad \text{and} \quad x_k = x_0 + kh, k = 1, 2, \dots, (2N-1).$$

Thus, the definite integral  $I$  can be written as

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

Applying Simpson's 1/3 rule as in equation

$$\int_{x_0}^{x_2} f(x)dx = x_0y_0 + x_1y_1 + x_2y_2 + \text{Error} = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi) \quad (iv)$$

to each of the integrals on the right-hand side of the above equation, we obtain

$$I = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \dots + (y_{2N-2} + 4y_{2N-1} + y_{2N})] - \frac{N}{90} h^5 y^{(iv)}(\xi)$$

That is

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \dots + y_{2N-1}) + 2(y_2 + y_4 + \dots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

This formula is called composite Simpson's 1/3 rule. The error term  $E$ , which is also called global error, is given by

$$E = -\frac{N}{90} h^5 y^{(iv)}(\xi) = -\frac{x_{2N} - x_0}{180} h^4 y^{(iv)}(\xi)$$

for some  $\xi$  in  $[x_0, x_{2N}]$ . Thus, in Simpson's 1/3 rule, the global error is of  $O(h^4)$ .

Similarly in deriving composite Simpson's 3/8 rule, we divide the interval of integration into  $n$  sub-intervals, where  $n$  is divisible by 3, and applying the integration formula

$$\int_{x_0}^{x_3} f(x)dx = \frac{3}{8} h(y_0 + 3y_1 + 3y_2 + y_3) - \frac{3}{80} h^5 y^{(iv)}(\xi)$$

to each of the integral given below

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)dx$$

We obtain the composite form of Simpson's 3/8 rule as

$$\int_a^b f(x)dx = \frac{h}{8} [y(a) + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y(b)]$$

With the global error  $E$  given by

$$E = -\frac{x_n - x_0}{80} h^4 y^{(iv)}(\xi)$$

It may be noted that the global error in Simpson's 1/3 and 3/8 rules are of the same order. However, if we consider the magnitudes of the error terms, we notice that Simpson's 1/3 rule is superior to Simpson's 3/8 rule. For illustration, we consider few examples.

### Example

Find the approximate value of  $y = \int_0^\pi \sin x dx$  using

(i) Trapezoidal rule

(ii) Simpson's 1/3 rule by dividing the range of integration into six equal parts. Calculate the percentage error from its true value in both the cases.

**Solution**

We shall at first divide the range of integration  $(0, \pi)$  into six equal parts so that each part is of width  $\pi/6$  and write down the table of values:

| X      | 0   | $\pi/6$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $5\pi/6$ | $\pi$ |
|--------|-----|---------|---------|---------|----------|----------|-------|
| Y=sinx | 0.0 | 0.5     | 0.8660  | 1.0     | 0.8660   | 0.5      | 0.0   |

Applying trapezoidal rule, we have

$$\int_0^\pi \sin x dx = \frac{h}{2} [y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

Here,  $h$ , the width of the interval is  $\pi/6$ . Therefore,

$$y = \int_0^\pi \sin x dx = \frac{\pi}{12} [0 + 0 + 2(3.732)] = \frac{3.1415}{6} \times 3.732 = 1.9540$$

Applying Simpson's 1/3 rule

$$\int_{x_0}^{x_2} f(x)dx = x_0 y_0 + x_1 y_1 + x_2 y_2 + \text{Error} = \frac{h}{3} (y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi)$$

We have

$$\begin{aligned} \int_0^\pi \sin x dx &= \frac{h}{3} [y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{\pi}{18} [0 + 0 + (4 \times 2) + (2)(1.732)] = \frac{3.1415}{18} \times 11.464 = 2.0008 \end{aligned}$$

But the actual value of the integral is

$$\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = 2$$

Hence, in the case of trapezoidal rule

The percentage of error

$$= \frac{2 - 1.954}{2} \times 100 = 23$$

While in the case of Simpson's rule the percentage error is

$$\frac{2 - 2.0008}{2} \times 100 = 0.04$$

(sign ignored)

**Example :**

From the following data, estimate the value of  $\int_1^5 \log x dx$  using Simpson's 1/3 rule. Also, obtain the value of h, so that the value of the integral will be accurate up to five decimal places.

| X      | 1.0    | 1.5    | 2.0    | 2.5    | 3.0    | 3.5    | 4.0    | 4.5    | 5.0    |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Y=logx | 0.0000 | 0.4055 | 0.6931 | 0.9163 | 1.0986 | 1.2528 | 1.3863 | 1.5041 | 1.6094 |

**Solution**

We have from the data,  $h = 0.5$ , and  $h = 0.5$ . Now using Simpson's 1/3 rule,

$$\begin{aligned} \int_1^5 \log x dx &= \frac{h}{3} [y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)] \\ &= \frac{0.5}{3} [(0 + 1.6094) + 4(0.4055 + 0.9163 + 1.2528 + 1.5041) + 2(0.6931 + 1.0986 + 1.3863)] \\ &= \frac{0.5}{3} (1.6094 + 16.3148 + 6.356) = 4.0467 \end{aligned}$$

The error in Simpson's rule is given by

$$E = -\frac{x_{2N} - x_0}{180} h^4 y^{(iv)}(\xi)$$

(ignoring the sign)

Since  $y = \log x, y' = \frac{1}{x}, y'' = -\frac{1}{x^2}, y''' = \frac{2}{x^3}, y^{(iv)} = -\frac{6}{x^4}$

$$\text{Max}_{1 \leq x \leq 5} y^{(iv)}(x) = 6,$$

$$\text{Max}_{1 \leq x \leq 5} y^{(iv)}(x) = 0.0096$$

Therefore, the error bounds are given by

$$\frac{(0.0096)(4)h^4}{180} < E < \frac{(6)(4)h^4}{180}$$

If the result is to be accurate up to five decimal places, then

$$\frac{24h^4}{180} < 10^{-5}$$

That is,  $h^4 < 0.000075$  or  $h < 0.09$ . It may be noted that the actual value of integrals is

$$\int_1^5 \log x dx = [x \log x - x]_1^5 = 5 \log 5 - 4$$

**Example :**

Evaluate the integral  $I = \int_0^1 \frac{dx}{1+x^2}$  by using

(i) Trapezoidal rule

(ii) Simpson's 1/3 rule by taking  $h = \frac{1}{4}$ . Hence compute the approximate value of  $\pi$ .

**Solution**

At first, we shall tabulate the function as

| X                     | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1      |
|-----------------------|---|---------------|---------------|---------------|--------|
| $y = \frac{1}{1+x^2}$ | 1 | 0.9412        | 0.8000        | 0.6400        | 0.5000 |

Using trapezoidal rule, and taking  $h = \frac{1}{4}$

$$I = \frac{h}{2} [y_0 + y_4 + 4(y_1 + y_2 + y_3)] = \frac{1}{8} [1.5 + 2(2.312)] = 0.7828$$

using Simpson's 1/3 rule, and taking  $h = \frac{1}{4}$ , we have

$$I = \frac{h}{3} [y_0 + y_4 + 4(y_1 + y_3) + 2y_2] = \frac{1}{12} [1.5 + 4(1.512) + 1.6] = 0.7854$$

But the closed form solution to the given integral is

$$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

Equating the last two equations, we get  $\pi = 3.1416$ .

**Example:** Compute the integral  $I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-x^2/2} dx$  using Simpson's 1/3 rule,

Taking  $h = 0.125$ .

**Solution** At the outset, we shall construct the table of the function as required.

| X                                       | 0      | 0.125  | 0.250  | 0.375  | 0.5    | 0.625  | 0.750  | 0.875  | 1.0    |
|-----------------------------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $y = \sqrt{\frac{2}{\pi}} \exp(-X^2/2)$ | 0.7979 | 0.7917 | 0.7733 | 0.7437 | 0.7041 | 0.6563 | 0.6023 | 0.5441 | 0.4839 |

Using Simpson's 1/3 rule, we have

$$\begin{aligned} &= \frac{h}{3} [y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)] \\ &= \frac{0.125}{3} [0.7979 + 0.4839 + 4(0.7917 + 0.7437 + 0.6563 + 0.5441) \\ &\quad + 2(0.7733 + 0.7041 + 0.6023)] \\ &= \frac{0.125}{3} (1.2812 + 10.9432 + 4.1594) \\ &= 0.6827 \end{aligned}$$

**Example :**

A missile is launched from a ground station. The acceleration during first 80 seconds of flight, as recorded, is given in the following table:

|            |       |       |       |       |       |       |       |       |       |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $t(s)$     | 0     | 10    | 20    | 30    | 40    | 50    | 60    | 70    | 80    |
| $a(m/s^2)$ | 30.00 | 31.63 | 33.34 | 35.47 | 37.75 | 40.33 | 43.25 | 46.69 | 50.67 |

Compute the velocity of the missile when  $t = 80$  s, using Simpson's 1/3 rule.

**Solution:**

Since acceleration is defined as the rate of change of velocity, we have

$$v = \int_0^{80} a \, dt \quad \text{Or} \quad \frac{dv}{dt} = a$$

Using Simpson's 1/3-rule, we have

$$\begin{aligned}
 v &= \frac{h}{3} [(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)] \\
 &= \frac{10}{3} [(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)] \\
 &= 3086.1 \, m/s
 \end{aligned}$$

Therefore, the required velocity is given by  $y = 3.0861$  km/s.

### Simpson's 3/8 rule

Consider the definite integral  $I = \int_a^b f(x)dx$

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2} (y_0 + y_1) + \text{Error} = \frac{h}{2} (y_0 + y_1) - \frac{h^3}{12} y''(\xi)$$

Then, if  $n = 2$ , the integration takes the form

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3} (y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi)$$

Thus Simpson's 1/3 rule is based on fitting three points with a quadratic.

Similarly, for  $n = 3$ , the integration is found to be

$$\int_{x_0}^{x_3} f(x)dx = \frac{h}{8} (y_0 + 3y_1 + 3y_2 + y_3) - \frac{h^5}{80} y^{(iv)}(\xi)$$

This is known as Simpson's 3/8 rule, which is based on fitting four points by a cubic. Still higher order Newton-Cotes integration formulae can be derived for large values of  $n$ .

### TRAPEZOIDAL RULE

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2} (y_0 + y_n) + E$$

### SIMPSON'S 1/3 RULE

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3} [y_0 + 4(y_1 + y_3 + \dots + y_{2N-1}) + 2(y_2 + y_4 + \dots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

$$E = -\frac{X_{2N} - X_0}{180} h^4 y^{(iv)}(\xi)$$

Simpson's 3/8 rule is

$$\int_a^b f(x)dx = \frac{3h}{8} [y(a) + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \dots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y(b)]$$

With the global error  $E$  given by

$$E = -\frac{X_n - X_0}{80} h^4 y^{(iv)}(\xi)$$

### ROMBERG'S INTEGRATION

We have observed that the trapezoidal rule of integration of a definite integral is of

$O(h^2)$ , while that of Simpson's 1/3 and 3/8 rules are of fourth-order accurate.

We can improve the accuracy of trapezoidal and Simpson's rules using Richardson's extrapolation procedure which is also called Romberg's integration method.

For example, the error in trapezoidal rule of a definite integral

$$I = \int_a^b f(x)dx$$

can be written in the form

$$I = I_T + c_1 h^2 + c_2 h^4 + c_3 h^6 + \dots$$

By applying Richardson's extrapolation procedure to trapezoidal rule, we obtain the following general formula

$$I_{Tm}\left(\frac{h}{2^m}\right) = \frac{4^m I_{T(m-1)}\left(\frac{h}{2^{m-1}}\right) - I_{T(m-1)}\left(\frac{h}{2^{m-1}}\right)}{4^{m-1}}$$

Where  $m = 1, 2, \dots$ , with

$$I_{T0}(h) = I_T(h).$$

For illustration, we consider the following example.

**Example:**

Using Romberg's integration method, find the value of  $\int_1^{1.8} y(x)dx$  starting with trapezoidal rule, for the tabular values

|          |       |       |       |       |       |       |       |       |       |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| x        | 1.0   | 1.1   | 1.2   | 1.3   | 1.4   | 1.5   | 1.6   | 1.7   | 1.8   |
| y = f(x) | 1.543 | 1.669 | 1.811 | 1.971 | 2.151 | 2.352 | 2.577 | 2.828 | 3.107 |

Solution:

$$x_0 = 1, \quad x_n = 1.8,$$

$$h = \frac{1.8-1.0}{N}, \quad x_i = x_0 + ih$$

Let  $I_T$  denote the integration by Trapezoidal rule, then for

$$N=1, h=0.8, I_T = \frac{h}{2} (y_0 + y_1)$$

$$= 0.4(1.543 + 3.107) = 1.8600$$

$$N=2, h=0.4, I_T = \frac{h}{2} (y_0 + 2y_1 + y_2) = 0.2[1.543 + 2(2.151) + 3.107]$$

$$= 1.7904$$

$$N=4, h=0.2, I_T = \frac{h}{2} [y_0 + 2(y_1 + y_2 + y_3) + y_4] = 0.1[1.543 + 2(1.811 + 2.151 + 2.577) + 3.107]$$

$$= 1.7728$$

Similarly for

$$N = 8, \quad h = 0.1,$$

$$I_T = 1.7684$$

Now, using Romberg's formula, we have

$$I_{T1}\left(\frac{h}{2}\right) = \frac{4(1.7904) - 1.8600}{3}$$

$$I_{T2}\left(\frac{h}{2^2}\right) = \frac{4^2(1.7728) - 1.7672}{4^2 - 1}$$

$$I_{T3}\left(\frac{h}{2^3}\right) = \frac{4^3(1.7672) - 1.77317}{4^3 - 1} = 1.7671$$

that the value of the tabulated integral is 1.7671.

## DOUBLE INTEGRATION

To evaluate numerically a double integral of the form

$$I = \int \left[ \int (x, y) dx \right] dy$$

over a rectangular region bounded by the lines  $x = a$ ,  $x = b$ ,  $y = c$ ,  $y = d$  we shall employ either trapezoidal rule or Simpson's rule, repeatedly With respect to one variable at a time. Noting that, both the integrations are just a linear combination of values of the given function at different values of the independent variable, we divide the interval  $[a, b]$  into  $N$  equal sub-intervals of size  $h$ , such that  $h = (b - a)/N$ ; and the interval  $(c, d)$  into  $M$  equal sub-intervals of size  $k$ , so that  $k = (d - c)/M$ . Thus, we have

$$x_i = x_0 + ih, \quad x_0 = a,$$

$$x_N = b, \quad \text{for } i = 1, 2, \dots, N-1$$

$$y_i = y_0 + ik, \quad y_0 = c,$$

$$y_M = d, \quad \text{for } i = 1, 2, \dots, M-1$$

Thus, we can generate a table of values of the integrand, and the above procedure of integration is illustrated by considering a couple of examples.

**Example** Evaluate the double integral

$$I = \int_1^2 \int_1^2 \frac{dx dy}{x + y}$$

by using trapezoidal rule, with  $h = k = 0.25$ .

**Solution**

Taking  $x = 1, 1.25, 1.50, 1.75, 2.0$  and  $y = 1, 1.25, 1.50, 1.75, 2.0$ , the following table is generated using the integrand

$$f(x, y) = \frac{1}{x + y}$$

| x    | y      |        |        |        |        |
|------|--------|--------|--------|--------|--------|
|      | 1.00   | 1.25   | 1.50   | 1.75   | 2.00   |
| 1.00 | 0.5    | 0.4444 | 0.4    | 0.3636 | 0.3333 |
| 1.25 | 0.4444 | 0.4    | 0.3636 | 0.3333 | 0.3077 |
| 1.50 | 0.4    | 0.3636 | 0.3333 | 0.3077 | 0.2857 |
| 1.75 | 0.3636 | 0.3333 | 0.307  | 0.2857 | 0.2667 |
| 2.00 | 0.3333 | 0.3077 | 0.2857 | 0.2667 | 0.25   |

Keeping one variable say  $x$  fixed and varying the variable  $y$ , the application of trapezoidal rule to each row in the above table gives

$$\int_1^2 f(1, y) dy = \frac{0.25}{2} [0.5 + 2(0.4444 + 0.4 + 0.3636) + 0.3333]$$

$$= 0.4062$$

$$\int_1^2 f(1.25, y) dy = \frac{0.25}{2} [0.4444 + 2(0.4 + 0.3636 + 0.3333) + 0.3077]$$

$$= 0.3682$$

$$\int_1^2 f(1.5, y) dy = \frac{0.25}{2} [0.4 + 2(0.3636 + 0.3333 + 0.3077) + 0.2857]$$

$$= 0.3369$$

$$\int_1^2 f(1.75, y) dy = \frac{0.25}{2} [0.3636 + 2(0.3333 + 0.3077 + 0.2857) + 0.2667]$$

$$= 0.3105$$

and

$$\int_1^2 f(2, y) dy = \frac{0.25}{2} [0.3333 + 2(0.3077 + 0.2857) + 0.25]$$

$$= 0.2879$$

Therefore

$$I = \int_1^2 \int_1^2 \frac{dxdy}{x+y} = \frac{h}{2} \{ f(1, y) + 2[f(1.25, y) + f(1.5, y) + f(1.75, y)] + f(2, y) \}$$

By use of the last equations we get the required result as

$$I = \frac{0.25}{2} [0.04602 + 2(0.3682 + 0.3369 + 0.3105) + 0.2879] = 0.3407$$

**Example :** Evaluate

$$\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\sin(x+y)} dxdy$$

by numerical double integration.

**Solution**

Taking  $x = y = \pi/4, 3\pi/8, \pi/2$ , we can generate the following table of the integrand

$$f(x, y) = \sqrt{\sin(x+y)}$$

| x        | y      |         |         |          |         |
|----------|--------|---------|---------|----------|---------|
|          | 0      | $\pi/8$ | $\pi/4$ | $3\pi/8$ | $\pi/2$ |
| 0        | 0.0    | 0.6186  | 0.8409  | 0.9612   | 1.0     |
| $\pi/8$  | 0.6186 | 0.8409  | 0.9612  | 1.0      | 0.9612  |
| $\pi/4$  | 0.8409 | 0.9612  | 1.0     | 0.9612   | 0.8409  |
| $3\pi/8$ | 0.9612 | 1.0     | 0.9612  | 0.8409   | 0.6186  |
| $\pi/2$  | 1.0    | 0.9612  | 0.8409  | 0.6186   | 0.0     |

Keeping one variable as say  $x$  fixed and  $y$  as variable, and applying trapezoidal rule to each row of the above table, we get

$$\int_0^{\pi/2} f(0, y) dy = \frac{\pi}{16} [0.0 + 2(0.6186 + 0.8409 + 0.9612) + 1.0] = 1.1469$$

$$\int_0^{\pi/2} f\left(\frac{\pi}{8}, y\right) dy = \frac{\pi}{16} [0.6186 + 2(0.8409 + 0.9612 + 1.0) + 0.9612] = 1.4106$$

Similarly we get

$$\int_0^{\pi/2} f\left(\frac{\pi}{4}, y\right) dy = 1.4778,$$

$$\int_0^{\pi/2} f\left(\frac{3\pi}{8}, y\right) dy = 1.4106$$

and

$$\int_0^{\pi/2} f\left(\frac{\pi}{2}, y\right) dy = 1.1469$$

Using these results, we finally obtain

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin(x+y)}{\sqrt{x^2+y^2}} dx dy = \frac{\pi}{16} \left\{ f(0, y) + 2 \left[ f\left(\frac{\pi}{8}, y\right) + f\left(\frac{\pi}{4}, y\right) + f\left(\frac{3\pi}{8}, y\right) + f\left(\frac{\pi}{2}, y\right) \right] \right\}$$

$$= \frac{\pi}{16} [1.1469 + 2(1.4106 + 1.4778 + 1.4106) + 1.1469]$$

$$= 2.1386$$

## Ordinary Differential Equations

### Introduction

Taylor Series

Euler Method

Runge-Kutta Method

Predictor Corrector Method

### Introduction

Many problems in science and engineering when formulated mathematically are readily expressed in terms of ordinary differential equations (ODE) with initial and boundary condition.

### Example

The trajectory of a ballistic missile, the motion of an artificial satellite in its orbit, are governed by ordinary differential equations.

Theories concerning electrical networks, bending of beams, stability of aircraft, etc., are modeled by differential equations.

To be more precise, the rate of change of any quantity with respect to another can be modeled by an ODE

Closed form solutions may not be possible to obtain, for every modeled problem, while numerical methods exist, to solve them using computers.

In general, a linear or non-linear ordinary differential equation can be written as

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \dots, \frac{d^{n-1}y}{dt^{n-1}}\right)$$

Here we shall focus on a system of first order

differential equations of the form  $\frac{dy}{dt} = f(t, y)$  with the initial condition  $y(t_0) = y_0$ ,

which is called an initial value problem (IVP).

It is justified, in view of the fact that any higher order ODE can be reduced to a system of first order differential equations by substitution.

For example, consider a second order differential equation of the form

$$y'' = f(t, y, y')$$

Introducing the substitution  $p = y'$ , the above equation reduces to a system of two first order differential equations, such as

$$y' = p, \quad p' = f(t, y, p)$$

### Theorem

Let  $f(t, y)$  be real and continuous in the strip  $R$ , defined by  $t \in [t_0, T]$ ,  $-\infty \leq y \leq \infty$

Then for any  $t \in [t_0, T]$  and for any  $y_1, y_2$ , there exists a constant  $L$ , satisfying the

inequality  $|f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2|$  so that  $|f_y(t, y)| \leq L$ , for every  $t, y \in R$

Here,  $L$  is called *Lipschitz constant*.

If the above conditions are satisfied, then for any  $y_0$ , the IVP has a unique solution  $y(t)$ ,

for  $t \in [t_0, T]$

In fact, we assume the existence and uniqueness of the solution to the above IVP

The function may be linear or non-linear. We also assume that the function  $f(t, y)$  is sufficiently differentiable with respect to either  $t$  or  $y$ .

## TAYLOR'S SERIES METHOD

Consider an initial value problem described by

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

Here, we assume that  $f(t, y)$  is sufficiently differentiable with respect to  $x$  and  $y$ .

If  $y(t)$  is the exact solution, we can expand  $y(t)$  by Taylor's series about the point  $t = t_0$

and obtain

$$y(t) = y(t_0) + (t - t_0)y'(t_0) + \frac{(t - t_0)^2}{2!}y''(t_0) + \frac{(t - t_0)^3}{3!}y'''(t_0) + \frac{(t - t_0)^4}{4!}y^{IV}(t_0) + \dots$$

Since, the solution is not known, the derivatives in the above expansion are not known explicitly. However,  $f$  is assumed to be sufficiently differentiable and therefore, the derivatives can be obtained directly from the given differential equation.

Noting that  $f$  is an implicit function of  $y$ , we have

$$\left. \begin{aligned} y' &= f(t, y) \\ y'' &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = f_x + f_y y' \end{aligned} \right\}$$

Similarly

$$\begin{aligned}
 y''' &= f_{xx} + ff_{xy} + f(f_{xy} + ff_{yy}) + f_y(f_x + ff_y) \\
 &= f + 2ff + f^2f + f(f + ff) \\
 y^{IV} &= f^{xx} + 3ff^{xy} + 3f^2f^{yy} \\
 &\quad + f(f^{xxx} + 2ff^{xxy} + f^2f^{xyy}) \\
 &\quad + 3(f^y + ff^{xx})(ff^{xy} + ff^{yy}) \\
 &\quad + f^2(f^x + ff^y)^{xy}
 \end{aligned}$$

Continuing in this manner, we can express any derivative of  $y$  in terms of  $f(t, y)$  and its partial derivatives.

### Example

Using Taylor's series method, find the solution of the initial value problem

$$\frac{dy}{dt} = t + y, \quad y(1) = 0$$

at  $t = 1.2$ , with  $h = 0.1$  and compare the result with the closed form solution

### Solution

Let us compute the first few derivatives from the given differential equation as follows:

$$y' = t + y, \quad y'' = 1 + y', \quad y''' = y'', \\ y^{IV} = y''', \quad y^V = y^{IV}$$

Prescribing the initial condition, that is, at

$$t_0 = 1, y_0 - y(t_0) = 0, \text{ we have}$$

$$y'_0 = 1, y''_0 = 2, \\ y'''_0 = y^{IV}_0 = y^V_0 = 2$$

Now, using Taylor's series method, we have

$$y(t) = y_0 + (t - t_0)y'_0 + \frac{(t - t_0)^2}{2}y''_0 + \frac{(t - t_0)^3}{6}y'''_0 \\ + \frac{(t - t_0)^4}{24}y^{IV}_0 + \frac{(t - t_0)^5}{120}y^V_0 + \dots$$

Substituting the above values of the derivatives, and the initial condition, we obtain

$$y(1.1) = 0 + (0.1)(1) + \frac{0.01}{2}(2) + \frac{0.001}{6}(2) + \frac{0.00001}{120}(2) + \dots \\ = 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12} + \frac{0.00001}{60} + \dots \\ = 0.1 + 0.01 + 0.000333 + 0.0000083 + 0.0000001 + \dots \\ = 0.1103414$$

Therefore

$$y(1.1) = y_1 = 0.1103414 \cong 0.1103$$

Taking  $y_1 = 0.1103$  at  $t = 1.1$ , the values of the derivatives are

$$y'_1 = 1.1 + 0.1103 = 1.2103 \\ y''_1 = 1 + 1.2103 = 2.2103 \\ y'''_1 = y^{IV}_1 = y^V_1 = 2.2103$$

Substituting the value of  $y_1$  and its derivatives into Taylor's series expansion we get,

after retaining terms up to fifth derivative only.....

$$y(1.2) = y_1 + (t - t_1)y'_1 + \frac{(t - t_1)^2}{2}y''_1 \\ + \frac{(t - t_1)^3}{6}y'''_1 + \frac{(t - t_1)^4}{24}y^{IV}_1 + \frac{(t - t_1)^5}{120}y^V_1 \\ y(1.2) = 0.1103 + 0.12103 \\ + 0.0110515 + 0.0003683 \\ + 0.000184 + 0.0000003 \\ = 0.2429341 \sim 0.2429 \approx 0.243$$

To obtain the closed form solution, we rewrite the given IVP as

$$\frac{dy}{dt} - y = t \quad \text{or} \quad d(ye^{-t}) = te^{-t}$$

On integration, we get

$$y = -e^t (te^{-t} + e^{-t}) + ce^t$$

$$= ce^t - t - 1$$

Using the initial condition, we get Therefore, the closed form solution is

$$y = -t - 1 + 2e^{t-1}$$

When  $t = 1.2$ , the closed form solution becomes

$$y(1.2) = -1.2 - 1 + 2(1.2214028)$$

$$= -2.2 + 2.4428056$$

$$= 0.2428 \approx 0.243$$

### Example

Using Taylor's Series method taking algorithm of order 3, solve the initial value problem

$$y' = 1 - y \quad ; \quad y(0) = 0 \quad \text{with } h = 0.25 \quad \text{at } x = .50$$

**Solution:**

Let us compute the first three derivatives

$$y' = 1 - y \quad y'' = -y' \quad y''' = -y''$$

The initial condition is

$$t_0 = 0 \quad y_0 = 0, \text{ we have}$$

$$y' = 1 - (0) = 1$$

$$y'' = -y' = -1$$

$$y''' = -y'' = -(-1) = 1$$

Now, Taylor's series method algorithm is

$$y(t_1) = y_0 + (t_1 - t_0)y'_0 + \frac{(t_1 - t_0)^2}{2}y''_0 + \frac{(t_1 - t_0)^3}{3!}y'''_0$$

$$\text{OR } y(t_1) = y_0 + hy'_0 + h^2 \frac{1}{2}y''_0 + h^3 \frac{1}{3!}y'''_0 \quad h = 0.25$$

$$y(0.25) = 0 + (0.25)(1) + \frac{(0.25)^2}{2}(-1) + \frac{(0.25)^3}{3!}(1)$$

$$y(0.25) = 0.25 - \frac{0.0625}{2} + \frac{0.015625}{6}$$

$$y(0.25) = 0.25 - 0.03125 + 0.002604167$$

$$y(0.25) = 0.22135$$

Taking  $y_1 = 0.22135$ , Now

$$y' = 1 - y = 1 - (0.22135) = 0.7786$$

$$y'' = -y' = -0.7786$$

$$y''' = -y'' = -(-0.7786) = 0.7786$$

$$y(t_2) = y_1 + (t_2 - t_1)y'_1 + \frac{(t_2 - t_1)^2}{2}y''_1 + \frac{(t_2 - t_1)^3}{3!}y'''_1$$

$$\text{OR } y(t_2) = y_1 + hy'_1 + h^2 \frac{1}{2}y''_1 + \frac{1}{6}h^3y'''_1 \quad h = .25$$

$$= 0.22135 + (0.25)(0.7786) + \frac{1}{2}(0.25)^2(-0.7786) + \frac{1}{6}(0.25)^3(0.7786)$$

$$= 0.22135 + 0.19465 - 0.02433 + 0.002027$$

$$= 0.39369$$

## EULER METHOD

Euler method is one of the oldest numerical methods used for integrating the ordinary differential equations. Though this method is not used in practice, its understanding will help us to gain insight into nature of predictor-corrector method

Consider the differential equation of first order with the initial condition  $y(t_0) = y_0$ .

$$\frac{dy}{dt} = f(t, y)$$

The integral of this equation is a curve in,  $ty$ -plane.

Here, we find successively  $y_1, y_2, \dots, y_m$ ; where  $y_m$  is the value of  $y$  at  $t = t_0 + mh$ ,

$m = 1, 2, \dots$  and  $h$  being small.

Here we use a property that in a small interval, a curve is nearly a straight line. Thus at  $(t_0, y_0)$ , we approximate the curve by a tangent at that point.

Therefore,

$$\left( \frac{dy}{dt} \right)_{(t_0, y_0)} = \frac{y - y_0}{t - t_0} = f(t_0, y_0)$$

That is

$$y = y_0 + (t - t_0) f(t_0, y_0)$$

Hence, the value of  $y$  corresponding to  $t = t_1$  is given by

$$y_1 = y_0 + (t_1 - t_0) f(t_0, y_0)$$

Similarly approximating the solution curve in the next interval  $(t_1, t_2)$  by a line through

$(t_1, y_1)$  having its slope  $f(t_1, y_1)$ , we obtain

$$y_2 = y_1 + hf(t_1, y_1)$$

Thus, we obtain in general, the solution of the given differential equation in the form of a recurrence relation

$$y_{m+1} = y_m + hf(t_m, y_m)$$

Geometrically, this method has a very simple meaning. The desired function curve is approximated by a polygon train, where the direction of each part is determined by the value of the function  $f(t, y)$  at its starting point.

### Example

Given  $\frac{dy}{dt} = \frac{y-t}{y+t}$  with the initial condition  $y = 1$  at  $t = 0$ . Using Euler method, find  $y$

approximately at  $x = 0.1$ , in five steps.

### Solution

Since the number of steps are five, we shall proceed in steps of  $(0.1)/5 = 0.02$ .

Therefore, taking step size

$h = 0.02$ , we shall compute the value of  $y$  at

$t = 0.02, 0.04, 0.06, 0.08$  and  $0.1$

Thus  $y_1 = y_0 + hf(t_0, y_0)$ , where  $y_0 = 1, t_0 = 0$

Therefore  $y_1 = 1 + 0.02 \frac{1-0}{1+0} = 1.02$

$$y_2 = y_1 + hf(t_1, y_1) = 1.02 + 0.02 \frac{1.02 - 0.02}{1.02 + 0.02} = 1.0392$$

$$y_3 = y_2 + hf(t_2, y_2) = 1.0392 + 0.02 \frac{1.0392 - 0.04}{1.0392 + 0.04} = 1.0577$$

Similarly,

$$y_4 = y_3 + hf(t_3, y_3) = 1.0577 + 0.02 \frac{1.0392 + 0.04}{1.0577 - 0.06} = 1.0738$$

$$y_5 = y_4 + hf(t_4, y_4) = 1.0738 + 0.02 \frac{1.0577 + 0.06}{1.0738 - 0.08} = 1.0910$$

Hence the value of  $y$  corresponding to  $t = 0.1$  is 1.091

### Example

Solve the differential equation

$$y' = x + y \quad ; \quad y(0) = 1$$

in the interval  $[0, 0.5]$  using Euler's method by taking  $h=0.1$

**Soution:**

|       |     |
|-------|-----|
| $x_0$ | 0   |
| $x_1$ | 0.1 |
| $x_2$ | 0.2 |
| $x_3$ | 0.3 |
| $x_4$ | 0.4 |
| $x_5$ | 0.5 |

Since  $h = 0.1$  so we calculate values at  $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5$

$$y(0) = 1 \text{ here } x_0 = 0 \quad y_0 = 1$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 1 + (0.1)(0+1) = 1 + 0.1 = 1.1$$

$$y_1 = 1.1 \quad x_1 = 0.1$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = 1.1 + (0.1)(0.1+1.1) = 1.1 + 0.12 = 1.22$$

now

$$y_2 = 1.22 \quad x_2 = 0.2$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$y_3 = 1.22 + (0.1)(1.22 + 0.2) = 1.22 + 0.142 = 1.3620$$

now

$$y_3 = 1.3620 \quad x_3 = 0.3$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$y_4 = 1.3620 + (0.1)(1.3620 + 0.3) = 1.3620 + 0.1662 = 1.5282$$

now

$$y_4 = 1.5282 \quad x_4 = 0.4$$

$$y_5 = y_4 + hf(x_4, y_4)$$

$$y_5 = 1.5282 + (0.1)(1.5282 + 0.4) = 1.7210$$

now

$$y_5 = 1.7210 \quad x_5 = 0.5$$

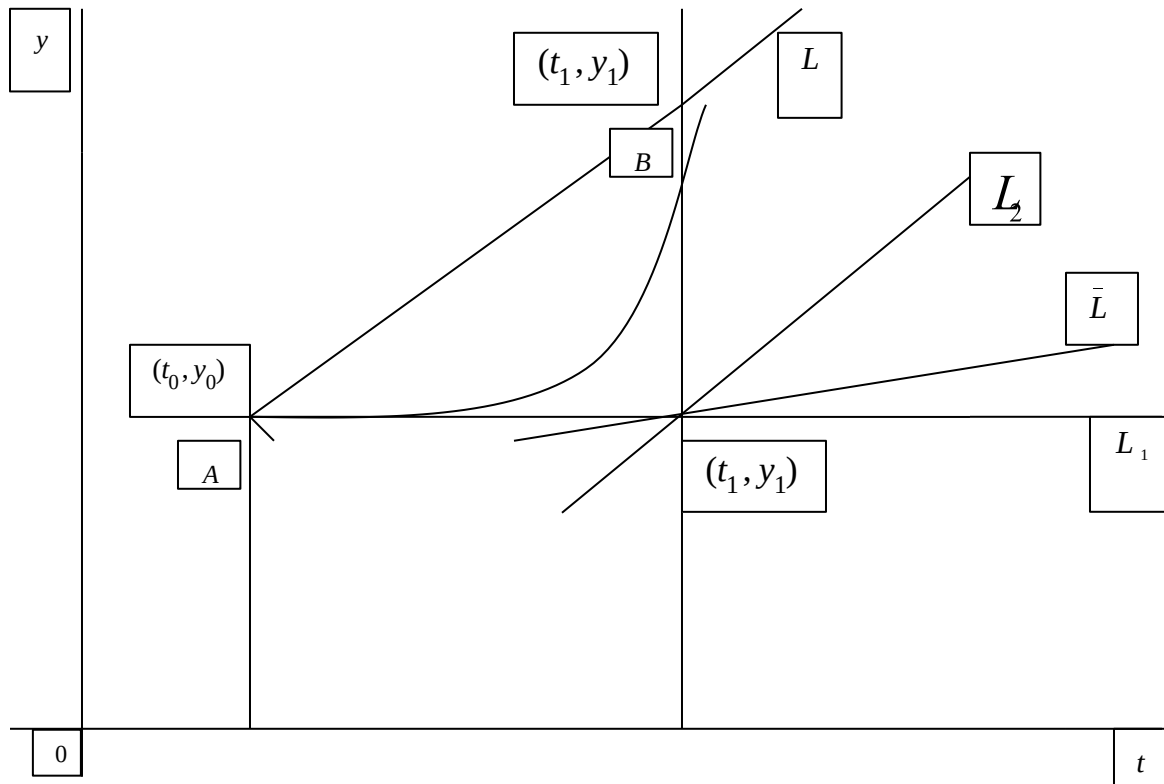
$$y_6 = y_5 + hf(x_5, y_5)$$

$$y_6 = 1.7210 + (0.1)(1.7210 + 0.5) = 1.9431$$

Hence the value of  $y$  corresponding to  $t = 0.5$  is 1.9431

### MODIFIED EULER'S METHOD

The modified Euler's method gives greater improvement in accuracy over the original Euler's method. Here, the core idea is that we use a line through  $(t_0, y_0)$  whose slope is the average of the slopes at  $(t_0, y_0)$  and  $(t_1, y_1^{(1)})$  Where  $y_1^{(1)} = y_0 + hf(t_0, y_0)$  is the value of  $y$  at  $t = t_1$  as obtained in Euler's method, which approximates the curve in the interval  $(t_0, t_1)$



(Modified Euler's Method)

Geometrically, from Figure, if  $L_1$  is the tangent at  $(t_1, y_1)$ ,  $L_2$  is the line through  $(t_1, y_1)$  of slope  $f(t_1, y_1^{(1)})$  and  $\bar{L}$  is the line through  $(t_1, y_1^{(1)})$  but with a slope equal to the average of  $f(t_0, y_0)$  and  $f(t_1, y_1^{(1)})$ ,.... the line  $L$  through  $(t_0, y_0)$  and parallel to  $L_1$  is used to approximate the curve in the interval  $(t_0, t_1)$ .

Thus, the ordinate of the point  $B$  will give the value of  $y_1$ .

Now, the equation of the line  $AL$  is given by

$$\begin{aligned} y_1 &= y_0 + \left[ \frac{f(t_0, y_0) + f(t_1, y_1^{(1)})}{2} \right] (t_1 - t_0) \\ &= y_0 + h \left[ \frac{f(t_0, y_0) + f(t_1, y_1^{(1)})}{2} \right] \end{aligned}$$

Similarly proceeding, we arrive at the recurrence relation

$$y_{m+1} = y_m + h \left[ \frac{f(t_m, y_m) + f(t_{m+1}, y_{m+1}^{(1)})}{2} \right]$$

This is the *modified Euler's method*.

### Example

Using modified Euler's method, obtain the solution of the differential equation

$$\frac{dy}{dt} = t + \sqrt{y} = f(t, y)$$

with the initial condition

$$y_0 = 1 \text{ at } t_0 = 0 \text{ for the range } 0 \leq t \leq 0.6 \text{ in steps of } 0.2$$

### Solution

At first, we use Euler's method to get

$$y_1^{(1)} = y_0 + hf(t_0, y_0) = 1 + 0.2(0 + 1) = 1.2$$

Then, we use modified Euler's method to find

$$\begin{aligned} y(0.2) = y_1 = y_0 + h \frac{f(t_0, y_0) + f(t_1, y_1^{(1)})}{2} \\ = 1.0 + 0.2 \frac{1 + (0.2 + \sqrt{1.2})}{2} = 1.2295 \end{aligned}$$

Similarly proceeding, we have from Euler's method

$$\begin{aligned} y_2^{(1)} = y_1 + hf(t_1, y_1) &= 1.2295 + 0.2(0.2 + \sqrt{1.2295}) \\ &= 1.4913 \end{aligned}$$

Using modified Euler's method, we get

$$\begin{aligned} y_2 = y_1 + h \frac{f(t_1, y_1) + f(t_2, y_2^{(1)})}{2} \\ = 1.2295 + 0.2 \frac{(0.2 + \sqrt{1.2295}) + (0.4 + \sqrt{1.4913})}{2} \\ = 1.5225 \end{aligned}$$

Finally,

$$\begin{aligned} y_3^{(1)} = y_2 + hf(t_2, y_2) &= 1.5225 + 0.2(0.4 + \sqrt{1.5225}) \\ &= 1.8493 \end{aligned}$$

Modified Euler's method gives

$$\begin{aligned} y(0.6) = y_3 = y_2 + h \frac{f(t_2, y_2) + f(t_3, y_3^{(1)})}{2} \\ = 1.5225 + 0.1 \left[ (0.4 + \sqrt{1.5225}) + (0.6 + \sqrt{1.8493}) \right] \\ = 1.8819 \end{aligned}$$

Hence, the solution to the given problem is given by

|   |        |        |        |
|---|--------|--------|--------|
| t | 0.2    | 0.4    | 0.6    |
| y | 1.2295 | 1.5225 | 1.8819 |

## RUNGE – KUTTA METHOD

These are computationally, most efficient methods in terms of accuracy. They were developed by two German mathematicians, Runge and Kutta.

They are distinguished by their orders in the sense that they agree with Taylor's series

solution up to terms of  $h^r$ , where  $r$  is the order of the method. These methods do not demand prior computation of higher derivatives of  $y(t)$  as in TSM.

Fourth-order Runge-Kutta methods are widely used for finding the numerical solutions of linear or non-linear ordinary differential equations, the development of which is complicated algebraically.

Therefore, we convey the basic idea of these methods by developing the second-order Runge-Kutta method which we shall refer hereafter as R-K method.

Please recall that the modified Euler's method: which can be viewed as

$$y_{n+1} = y_n + h \quad (\text{average of slopes})$$

This, in fact, is the basic idea of R-K method.

Here, we find the slopes not only at  $t$  but also at several other interior points, and take the

weighted average of these slopes and add to  $y_n$  to get  $y_{n+1}$ . Now, we shall derive the

second order R-K method in the following slides.

Consider the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_n) = y_n$$

We also define

$$k_1 = hf(t_n, y_n), \quad k_2 = hf(t_n + \alpha h, y_n + \beta k_1)$$

and take the weighted average of  $k_1$  and  $k_2$  and add to  $y_n$  to get  $y_{n+1}$

We seek a formula of the form  $y_{n+1} = y_n + W_1 k_1 + W_2 k_2$

Where  $\alpha, \beta, W_1$  and  $W_2$  are constants to be determined so that the above equation agree with the Taylor's series expansion as high an order as possible

Thus, using Taylor's series expansion, we have

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(t_n) + \dots$$

Rewriting the derivatives of  $y$  in terms of  $f$  of the above equation, we get

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2} (f_t + ff_y) + \frac{h^3}{6} \left[ f_{tt} + 2ff_{ty} + \frac{f^2}{2} + f_{yy} + f_y(f_t + ff_y) \right] + O(h^4)$$

Here, all derivatives are evaluated at  $(t_n, y_n)$ .

Next, we shall rewrite the given equation after inserting the expressions of  $k_1$  and  $k_2$  as

$$y_{n+1} = y_n + W_1 hf(t_n, y_n) + W_2 hf(t_n + \alpha h, y_n + \beta k_1)$$

Now using Taylor's series expansion of two variables, we obtain

$$y_{n+1} = y_n + W_1 h f(t_n, y_n) + W_2 h \left[ f(t_n, y_n) + (\alpha h f_t + \beta k_1 f_y) \right. \\ \left. \left( \frac{\alpha^2 h^2}{2} f_{tt} + \alpha \beta h^2 f_{ty} + \frac{\beta^2 k_1^2}{2} f_{yy} \right) + O(h^3) \right]$$

Here again, all derivatives are computed at  $(t_n, y_n)$ . On inserting the expression for  $k_1$ , the above equation becomes

$$y_{n+1} = y_n + (W_1 + W_2) h f + W_2 h \left[ (\alpha h f_t + \beta h f_y) \right. \\ \left. + \left( \frac{\alpha^2 h^2}{2} f_{tt} + \alpha \beta h^2 f_{ty} + \frac{\beta^2 h^2}{2} f_{yy} \right) + O(h^3) \right]$$

On rearranging in the increasing powers of  $h$ , we get

$$y_{n+1} = y_n + (W_1 + W_2) h f + W_2 h^2 (\alpha f_t + \beta f_y) \\ + W_2 h^3 \left( \frac{\alpha^2}{2} f_{tt} + \alpha \beta f_{ty} + \frac{\beta^2}{2} f_{yy} \right) + O(h^4)$$

Now, equating coefficients of  $h$  and  $h^2$  in the two equations, we obtain

$$W_1 + W_2 = 1, \quad W_2 (\alpha f_t + \beta f_y) = \frac{f_t + f_y}{2}$$

Implying

$$W_1 + W_2 = 1, \quad W_2 \alpha = W_2 \beta = \frac{1}{2}$$

Thus, we have three equations in four unknowns and so, we can choose one value arbitrarily. Solving we get

$$W_1 = 1 - W_2, \quad \alpha = \frac{1}{2W_2}, \quad \beta = \frac{1}{2W_2}$$

Where  $W_2$  is arbitrary and various values can be assigned to it.

We now consider two cases, which are popular

### **Case I**

If we choose  $W_2 = 1/3$ ,

then

$$W_1 = 2/3 \text{ and } \alpha = \beta = 3/2 \\ y_{n+1} = y_n + \frac{1}{3} (2k_1 + k_2) \\ k_1 = hf(t, y), \quad k_2 = hf \left( t + \frac{3}{2} h, y + \frac{3}{2} k_1 \right)$$

**Case II:** If we consider

$$W_2 = 1/2, \text{ then } W_1 = 1/2 \text{ and } \alpha = \beta = 1.$$

$$\text{Then } y_{n+1} = y_n + \frac{k_1 + k_2}{2}$$

$$k_1 = hf(t, y), \quad k_2 = hf(t + h, y + k_1)$$

In fact, we can recognize that this equation is the modified Euler's method and is therefore a special case of a 2<sup>nd</sup> order Runge-Kutta method. These equations are known as 2<sup>nd</sup> order R-K Methods, since they agree with Taylor's series solution up to the term  $\frac{h^2}{2}$ .

Defining the local truncation error, TE, as the difference between the exact solution  $y(t_{n+1})$  at  $t = t_{n+1}$  and the numerical solution  $y_{n+1}$  obtained using the second order R-K method, we have

$$TE = y(t_{n+1}) - y_{n+1}$$

Now, substituting

$$W_2 = \frac{1}{2\alpha}, \quad W_1 = 1 - \frac{1}{2\alpha}, \quad \beta = \alpha,$$

into the above equation, we get

$$y_{n+1} = y_n + hf_{t_n} + \frac{h^2}{2} (f_{t_n} + ff_{ty})_{t=t_n} + \frac{h^3\alpha}{4} (f_{t_n} + 2ff_{ty} + f^2 f_{yy})_{t=t_n} + \dots$$

Finally we obtain

$$TE = h^3 \left[ \left( \frac{1-\alpha}{6} - \frac{\alpha}{4} \right) (f_{t_n} + 2ff_{ty} + f^2 f_{yy}) + \frac{1}{6} f_{t_n} (f_{t_n} + ff_{ty}) \right]$$

The expression can further be simplified to

$$TE = h^3 \left[ \left( \frac{1-\alpha}{6} - \frac{\alpha}{4} \right) (y''' - f y') + \frac{1}{6} f y' \right]$$

Therefore, the expression for local truncation error is given by

$$TE = h^3 \left[ \left( \frac{1-\alpha}{6} - \frac{\alpha}{4} \right) y''' + \frac{\alpha}{4} f y' \right]$$

Please verify that the magnitude of the TE in case I is less than that of case II

Following similar procedure, Runge-Kutta formulae of any order can be obtained.

However, their derivations becomes exceedingly lengthy and complicated.

Amongst them, the most popular and commonly used in practice is the R-K method of fourth-order, which agrees with the Taylor series method up to terms of  $O(h^4)$ .

This well-known fourth-order R-K method is described in the following steps.

$$y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}
 k_1 &= hf(t_n, y_n) \\
 k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
 k_3 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
 k_4 &= hf(t_n + h, y_n + k_3)
 \end{aligned}$$

Please note that the second-order Runge-Kutta method described above requires the evaluation of the function twice for each complete step of integration.

Similarly, fourth-order Runge-Kutta method requires the evaluation of the function four times. The discussion on optimal order R-K method is very interesting, but will be discussed some other time.

**Example**

Use the following second order Runge-Kutta method described by

$$y_{n+1} = y_n + \frac{1}{2}(2k_1 + k_2)$$

where  $k_1 = hf(x_n, y_n)$  and  $k_2 = hf\left(x_n + \frac{3}{2}h, y_n + \frac{3}{2}k_1\right)$

and find the numerical solution of the initial value problem described as

$$\frac{dy}{dx} = \frac{y+x}{y-x}, \quad y(0) = 1$$

at  $x = 0.4$  and taking  $h = 0.2$ .

**Solution**

Here

$$f(x, y) = \frac{y+x}{y-x}, \quad h = 0.2, \quad x_0 = 0, \quad y_0 = 1$$

We calculate  $k_1 = hf(x_0, y_0) = 0.2 \frac{1+0}{1-0} = 0.2$

$$k_2 = hf\left[x_0 + 0.3, y_0 + (1.5)(0.2)\right]$$

$$= hf(0.3, 1.3) = 0.2 \frac{1.3+0.3}{1.3-0.3} = 0.32$$

Now, using the given R-K method, we get

$$y(0.2) = y_1 = 1 + \frac{1}{2}(0.4 + 0.32) = 1.24$$

Now, taking  $x_1 = 0.2$ ,  $y_1 = 1.24$ , we calculate

$$k_1 = hf(x_1, y_1) = 0.2 \frac{1.24+0.2}{1.24-0.2} = 0.2769$$

$$k_2 = hf\left(x_1 + \frac{3}{2}h, y_1 + \frac{3}{2}k_1\right) = hf(0.5, 1.6554)$$

$$= 0.2 \frac{1.6554+0.5}{1.6554-0.5} = 0.3731$$

Again using the given R-K method, we obtain

$$y(0.4) = y_2 = 1.24 + \frac{1}{3}[2(0.2769) + 0.3731]$$

$$= 1.54897$$

**Example**

Solve the following differential equation

$$\frac{dy}{dt} = t + y$$

with the initial condition  $y(0) = 1$ , using fourth-order Runge-Kutta method from  $t = 0$  to  $t = 0.4$  taking  $h = 0.1$

**Solution**

The fourth-order Runge-Kutta method is described as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (1)$$

Where

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf(t_n + h, y_n + k_3) \end{aligned}$$

In this problem

$$f(t, y) = t + y, \quad h = 0.1, \quad t_0 = 0, \quad y_0 = 1.$$

As a first step, we calculate

$$\begin{aligned} k_1 &= hf(t_0, y_0) = 0.1(1) = 0.1 \\ k_2 &= hf(t_0 + 0.05, y_0 + 0.05) \\ &= hf(0.05, 1.05) = 0.1[0.05 + 1.05] \\ &= 0.11 \\ k_3 &= hf(t_0 + 0.05, y_0 + 0.055) \\ &= 0.1(0.05 + 1.055) \\ &= 0.1105 \\ k_4 &= 0.1(0.1 + 1.1105) = 0.12105 \end{aligned}$$

Now, we compute from

$$\begin{aligned} y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ &= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) \\ &= 1.11034 \end{aligned}$$

Therefore  $y(0.1) = y_1 = 1.1103$

In the second step, we have to find  $y_2 = y(0.2)$

We compute

$$\begin{aligned}
k_1 &= hf(t_1, y_1) = 0.1(0.1 + 1.11034) = 0.121034 \\
k_2 &= hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\
&= 0.1[0.15 + (1.11034 + 0.060517)] = 0.13208 \\
k_3 &= hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\
&= 0.1[0.15 + (1.11034 + 0.06604)] = 0.132638 \\
k_4 &= hf(t_1 + h, y_1 + k_3) \\
&= 0.1[0.2 + (1.11034 + 0.132638)] = 0.1442978
\end{aligned}$$

From Equation (1), we see that

$$\begin{aligned}
y_2 &= 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208) \\
&\quad + 2(0.132638) + 0.1442978] = 1.2428
\end{aligned}$$

Similarly we calculate,

$$\begin{aligned}
k_1 &= hf(t_2, y_2) = 0.1[0.2 + 1.2428] = 0.14428 \\
k_2 &= hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1[0.25 + (1.2428 + 0.07214)] = 0.156494 \\
k_3 &= hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1[0.3 + (1.2428 + 0.078247)] = 0.1571047 \\
k_4 &= hf(t_2 + h, y_2 + k_3) = 0.1[0.3 + (1.2428 + 0.1571047)] = 0.16999047
\end{aligned}$$

Using equation (1), we compute

$$y(0.3) = y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.399711$$

Finally, we calculate

$$\begin{aligned}
k_1 &= hf(t_3, y_3) = 0.1[0.3 + 1.3997] = 0.16997 \\
k_2 &= hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1[0.35 + (1.3997 + 0.084985)] = 0.1834685 \\
k_3 &= hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1[0.35 + (1.3997 + 0.091734)] = 0.1841434 \\
k_4 &= hf(t_3 + h, y_3 + k_3) = 0.1[0.4 + (1.3997 + 0.1841434)] = 0.19838434
\end{aligned}$$

Using them in equation (1), we get

$$\begin{aligned}
y(0.4) &= y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= 1.58363
\end{aligned}$$

which is the required result.

### Example

Apply Runge-Kutta Method of order four to solve the initial value problem at  $x=1.2$

$$y' = xy^{1/3} \quad ; \quad y(1) = 1 \quad ; \quad h = 0.1$$

**Solution:**

The Fourth-Order Runge-Kutta Method is described as :-

$$y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where,

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_n + h, y_n + k_3)$$

( Here, we are taking x as t )

**First Iteration:–**

$$t_0 = 1, \quad y_0 = 1$$

$$k_1 = hf(t_0, y_0) = 0.1 \left[ 1 \times 1^{1/3} \right] \\ = 0.1$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ = 0.1 \left[ (1 + 0.05) \times (1 + k_1/2)^{1/3} \right] \\ = 0.106721617$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ = 0.1 \left[ (1 + 0.05) \times (1 + k_2/2)^{1/3} \right] \\ = 0.10683536$$

$$k_4 = hf(t_n + h, y_n + k_3) \\ = 0.1 \left[ (1 + 0.1) \times (1 + k_3)^{1/3} \right] \\ = 0.113785527$$

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4) \\ y_1 = 1 + 0.10681658 \\ y_1 = 1.10681658$$

Second Iteration:–

$$t_1 = 1.1, \quad y_1 = 1.10681658$$

$$k_1 = hf(t_1, y_1) = 0.1 \left[ 1.1 \times 1.10681658^{1/3} \right] \\ = 0.113784884$$

$$k_2 = hf \left( t_n + \frac{h}{2}, y_n + \frac{k_1}{2} \right) \\ = 0.1 \left[ (1.1 + 0.05) \times (1.10681658 / 2)^{1/3} \right] \\ = 0.120961169$$

$$k_3 = hf \left( t_n + \frac{h}{2}, y_n + \frac{k_2}{2} \right) \\ = 0.1 \left[ (1.1 + 0.05) \times (1.10681658 + k_2 / 2)^{1/3} \right] \\ = 0.121085364$$

$$k_4 = hf(t_n + h, y_n + k_3) \\ = 0.1 \left[ (1.1 + 0.1) \times (1.10681658 + k_3)^{1/3} \right] \\ = 0.128499807$$

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4) \\ y_2 = 1.10681658 + 0.12106296 \\ y_2 = 1.22787954$$

So, at  $x = 1.2$ , we get:

$$y(1.2) = y_2 = 1.22787954$$

## RUNGE – KUTTA METHOD FOR SOLVING A SYSTEM OF EQUATIONS

The fourth-order Runge-Kutta method can be extended to numerically solve the higher-order ordinary differential equations- linear or non-linear

For illustration, let us consider a second order ordinary differential equation of the form

$$\frac{d^2 y}{dt^2} = f \left( t, y, \frac{dy}{dt} \right)$$

Using the substitution  $\frac{dy}{dt} = p$  this equation can be reduced to two first-order simultaneous differential equations, as given by

$$\frac{dy}{dt} = p = f_1(t, y, p), \quad \frac{dp}{dt} = f_2(t, y, p)$$

Now, we can directly write down the Runge-Kutta fourth-order formulae for solving the system.

Let the initial conditions of the above system be given by

$$y(x_n) = y_n, \quad y'(x_n) = p(x_n) = p_n$$

Then we define

$$\left. \begin{aligned} k_1 &= hf_1(t_n, y_n, p_n), & l_1 &= hf_2(t_n, y_n, p_n) \\ k_2 &= hf_1\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}, p_n + \frac{l_1}{2}\right), & l_2 &= hf_2\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}, p_n + \frac{l_1}{2}\right) \\ k_3 &= hf_1\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}, p_n + \frac{l_2}{2}\right), & l_3 &= hf_2\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}, p_n + \frac{l_2}{2}\right) \\ k_4 &= hf_1(t_n + h, y_n + k_3, p_n + l_3), & l_4 &= hf_2(t_n + h, y_n + k_3, p_n + l_3) \end{aligned} \right\}$$

Now, using the initial conditions  $y, p$  and 4<sup>th</sup>-order R-K formula, we compute

$$\left. \begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ p_{n+1} &= p_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \end{aligned} \right\}$$

This method can be extended on similar lines to solve system of  $n$  first order differential equations.

### Example

Solve the following equation

$$y'' - (0.1)(1 - y^2)y' + y = 0$$

Using 4<sup>th</sup> order Runge-Kutta method for  $x = 0.2$ , with the initial values  $y(0) = 1, y'(0) = 0$

**Solution:**

$$\text{Let } \frac{dy}{dx} = p = f_1(x, y, p)$$

$$\text{Then } \frac{dp}{dx} = (0.1)(1 - y^2)p - y = f_2(x, y, p)$$

Thus, the given equation reduced to two first-order equations.

In the present problem, we are given

$$x_0 = 0, y_0 = 1, p_0 = y'_0 = 0$$

Taking  $h = 0.2$ , we compute

$$\begin{aligned}
k_1 &= hf_1(x_0, y_0, p_0) = 0.2(0.0) = 0.0 \\
l_1 &= hf_2(x_0, y_0, p_0) = 0.2(0.0 - 1) = -0.2 \\
k_2 &= hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right) \\
&= hf_1(0.1, 1.0, -0.1) = -0.02 \\
l_2 &= hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right) \\
&= hf_2(0.1, 1.0, -0.1) = 0.2(-0.1) = -0.02 \\
k_3 &= hf_1\left(x_0 + h, y_0 + k_2, p_0 + l_2\right) \\
&= hf_1(0.1, 0.99, -0.1) = 0.2(-0.1) = -0.02 \\
l_3 &= hf_2\left(x_0 + h, y_0 + k_2, p_0 + l_2\right) = hf_2(0.1, 0.99, -0.1) \\
&= 0.2[(0.1)(0.0199)(-0.1) - 0.99] = -0.1980 \\
k_4 &= hf_1(x_0 + h, y_0 + k_3, p_0 + l_3) \\
&= hf_1(0.2, 0.98, -0.1980) = -0.0396 \\
l_4 &= hf_2(x_0 + h, y_0 + k_3, p_0 + l_3) = hf_2(0.2, 0.98, -0.1980) \\
&= 0.2[(0.1)(1 - 0.9604)(-0.1980) - 0.98] = -0.19616
\end{aligned}$$

Now,  $y(0.2) = y_1$  is given by

$$\begin{aligned}
y(0.2) &= y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\
&= 1 + \frac{1}{6}[0.0 + 2(-0.02) + 2(-0.02) + (-0.0396)] \\
&= 1 - 0.019935 = 0.9801 \\
y'(0.2) &= p_1 = p_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4] \\
&= 0 + \frac{1}{6}[-0.2 + 2(-0.02) + 2(-0.1980) + (-0.19616)] \\
&= -0.19869 (= -0.1987)
\end{aligned}$$

Therefore, the required solution is

$$y(0.2) = 0.9801, \quad y'(0.2) = -0.1987$$

### PREDICTOR – CORRECTOR METHOD

The methods presented so far are called single-step methods, where we have seen that the computation of  $y$  at  $t_{n+1}$  that is  $y_{n+1}$  requires the knowledge of  $y_n$  only.

In predictor-corrector methods which we will discuss now, is also known as multi-step methods.

To compute the value of  $y$  at  $t_{n+1}$ , we must know the solution  $y$  at  $t_n, t_{n-1}, t_{n-2}$ , etc.

Thus, a predictor formula is used to predict the value of  $y$  at  $t_{n+1}$  and then a corrector formula is used to improve the value of  $y_{n+1}$ .

Let us consider an IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_n) = y_n$$

Using simple Euler's and modified Euler's method, we can write down a simple predictor-corrector pair

(P – C) as

$$\begin{aligned} P: y_{n+1}^{(0)} &= y_n + hf(t_n, y_n) \\ C: y_{n+1}^{(1)} &= y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1}^{(0)})] \end{aligned}$$

Here,  $y_{n+1}^{(1)}$  is the first corrected value of  $y_{n+1}$ . The corrector formula may be

used iteratively as defined below:

$$y_{n+1}^{(r)} = y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1}^{(r-1)})], \quad r = 1, 2, \dots$$

The iteration is terminated when two successive iterates agree to the desired accuracy. In this pair, to extrapolate the value of  $y_{n+1}$ , we have approximated the solution curve in

the interval  $(t_n, t_{n+1})$  by a straight line passing through  $(t_n, y_n)$  and  $(t_{n+1}, y_{n+1})$ .

The accuracy of the predictor formula can be improved by considering a quadratic curve through the equally spaced points  $(t_{n-1}, y_{n-1}), (t_n, y_n), (t_{n+1}, y_{n+1})$ .

Suppose we fit a quadratic curve of the form

$$y = a + b(t - t_{n-1}) + c(t - t_n)(t - t_{n-1})$$

where  $a, b, c$  are constants to be determined. As the curve passes through  $(t_{n-1}, y_{n-1})$  and  $(t_n, y_n)$  and satisfies

$$\left( \frac{dy}{dt} \right)_{(t_n, y_n)} = f(t_n, y_n)$$

We obtain

$$y_{n-1} = a, \quad y_n = a + bh = y_{n-1} + bh$$

Therefore

$$b = \frac{y_n - y_{n-1}}{h}$$

and

$$\left(\frac{dy}{dt}\right)_{(t_n, y_n)} = f(t_n, y_n) = \{b + c[(t - t_{n-1}) + (t - t_n)]\}_{(t_n, y_n)}$$

Which give

$$f(t_n, y_n) = b + c(t_n - t_{n-1}) = b + ch$$

or

$$c = \frac{f(t_n, y_n) - (y_n - y_{n-1})}{h^2}$$

Substituting these values of  $a$ ,  $b$  and  $c$  into the quadratic equation, we get

$$y_{n+1} = y_{n-1} + 2(y_n - y_{n-1}) + 2[hf(t_n, y_n) - (y_n - y_{n-1})]$$

That is

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n)$$

Thus, instead of considering the P-C pair, we may consider the P-C pair given by

$$\left. \begin{array}{l} P: y_{n+1} = y_{n-1} + 2hf(t_n, y_n) \\ C: y_{n+1} = y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \end{array} \right\}$$

The essential difference between them is, the one given above is more accurate

However, this one can not be used to predict  $y_{n+1}$  for a given IVP, because its use

require the knowledge of past two points. In such a situation, a R-K method is generally used to start the predictor method.

### Milne's Method

It is also a multi-step method where we assume that the solution to the given IVP is known at the past four equally spaced point  $t_0, t_1, t_2$  and  $t_3$

To derive Milne's predictor-corrector pair, let us consider a typical IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

On integration between the limits  $t_0$  and  $t_4$ , we get

$$\begin{aligned} \int_{t_0}^{t_4} \frac{dy}{dt} dt &= \int_{t_0}^{t_4} f(t, y) dt \\ y_4 - y_0 &= \int_{t_0}^{t_4} f(t, y) dt \end{aligned}$$

But we know from Newton's forward difference formula

$$f(t, y) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2} \Delta^2 f_0 + \frac{s(s-1)(s-2)}{6} \Delta^3 f_0 + \dots$$

$$\begin{aligned} \text{where } s &= \frac{t-t_0}{h}, \quad t = t_0 + sh \\ y_4 &= y_0 + \int_{t_0}^{t_4} \left[ f_0 + s\Delta f_0 + \frac{s(s-1)}{2} \Delta^2 f_0 + \frac{s(s-1)(s-2)}{6} \Delta^3 f_0 \right. \\ &\quad \left. + \frac{s(s-1)(s-2)(s-3)}{24} \Delta^4 f_0 + \dots \right] dt \end{aligned}$$

Now, by changing the variable of integration (from  $t$  to  $s$ ), the limits of integration also changes (from 0 to 4), and thus the above expression becomes

$$y_4 = y_0 + h \int_0^4 \left[ 4f_0 + s\Delta f_0 + \frac{s(s-1)}{2} \Delta^2 f_0 + \frac{s(s-1)(s-2)}{6} \Delta^3 f_0 + \frac{s(s-1)(s-2)(s-3)}{24} \Delta^4 f_0 + \dots \right] ds$$

which simplifies to

$$y_4 = y_0 + h \left[ 4f_0 + 8\Delta f_0 + \frac{20}{3} \Delta^2 f_0 + \frac{8}{3} \Delta^3 f_0 + \frac{28}{90} \Delta^4 f_0 \right]$$

Substituting the differences

$$\Delta f_0 = f_1 - f_0, \Delta^2 f_0 = f_2 - 2f_1 + f_0,$$

It can be further simplified to

$$y_4 = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) + \frac{28}{90} h \Delta^4 f_0$$

Alternatively, it can also be written as

$$y_4 = y_0 + \frac{4h}{3} (2y'_1 - y'_2 + 2y'_3) + \frac{28}{90} h \Delta^4 y'_0$$

This is known as *Milne's predictor formula*.

Similarly, integrating the original over the interval  $t_0$  to  $t_2$  or  $s = 0$  to  $2$  and repeating the above steps, we get

$$y_2 = y_0 + \frac{h}{3} (y'_0 + 4y'_1 + y'_2) - \frac{1}{90} h \Delta^4 y'_0$$

which is known as *Milne's corrector formula*.

In general, Milne's predictor-corrector pair can be written as

$$\left. \begin{aligned} P: y_{n+1} &= y_{n-3} + \frac{4h}{3} (2y'_{n-2} - y'_{n-1} + 2y'_n) \\ C: y_{n+1} &= y_{n-1} + \frac{h}{3} (y'_{n-1} + 4y'_n + y'_{n+1}) \end{aligned} \right\}$$

From these equations, we observe that the magnitude of the truncation error in corrector formula is  $1/90 h \Delta^4 y'_0$  while the truncation error in predictor formula is  $28/90 h \Delta^4 y'_0$

Thus: TE in, c-formula is less than the TE in p-formula.

In order to apply this P – C method to solve numerically any initial value problem, we first predict the value of  $y_{n+1}$  by means of predictor formula, where derivatives are

computed using the given differential equation itself.

Using the predicted value  $y_{n+1}$ , we calculate the derivative  $y'_{n+1}$  from the given

differential equation and then we use the corrector formula of the pair to have the corrected value of  $y_{n+1}$ . This in turn may be used to obtain improved value of  $y_{n+1}$  by using corrector again. This in turn may be used to obtain improved value of  $y_{n+1}$  by

using the corrector again. This cycle is repeated until we achieve the required accuracy.

### Example

Find  $y(2.0)$  if  $y(t)$  is the solution of  $\frac{dy}{dt} = \frac{1}{2}(t+y)$

$$y(0) = 2, y(0.5) = 2.636,$$

$$y(1.0) = 3.595 \text{ and } y(1.5) = 4.968 \text{ Use Milne's P-C method.}$$

Solution

Taking  $t_0 = 0.0, t_1 = 0.5, t_2 = 1.0, t_3 = 1.5$   $y_0, y_1, y_2$  and  $y_3$  are given, we have to compute  $y_4$ , the solution of the given differential equation corresponding to  $t = 2.0$

The Milne's P – C pair is given as

$$P: y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$C: y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

From the given differential equation,  $y' = (t+y)/2$

We have

$$y'_1 = \frac{t_1 + y_1}{2} = \frac{0.5 + 2.636}{2} = 1.5680$$

$$y'_2 = \frac{t_2 + y_2}{2} = \frac{1.0 + 3.595}{2} = 2.2975$$

$$y'_3 = \frac{t_3 + y_3}{2} = \frac{1.5 + 4.968}{2} = 3.2340$$

Now, using predictor formula, we compute

$$\begin{aligned} y_4 &= y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) \\ &= 2 + \frac{4(0.5)}{3}[2(1.5680) - 2.2975 + 2(3.2340)] \\ &= 6.8710 \end{aligned}$$

Using this predicted value, we shall compute the improved value of  $y_4$  from corrector formula

$$y_4 = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$$

Using the available predicted value  $y_4$  and the initial values, we compute

$$y'_4 = \frac{t_4 + y_4}{2} = \frac{2 + 6.8710}{2} = 4.4355$$

$$y'_3 = \frac{t_3 + y_3}{2} = \frac{1.5 + 4.968}{2} = 3.2340$$

$$\text{and } y'_2 = 2.2975$$

Thus, the first corrected value of  $y_4$  is given by

$$y_4^{(1)} = 3.595 + \frac{0.5}{3} [2.2975 + 4(3.234) + 4.4355]$$

$$= 6.8731667$$

Suppose, we apply the corrector formula again, then we have

$$y_4^{(2)} = y_2 + \frac{h}{3} \left[ (y_2' + 4y_3' + (y_4^{(1)})') \right]$$

$$= 3.595 + \frac{0.5}{3} \left[ 2.2975 + 4(3.234) + \frac{2 + 6.8731667}{2} \right]$$

$$= 6.8733467$$

Finally,  $y(2.0) = y_4 = 6.8734$ .

### Example

Tabulate the solution of  $\frac{dy}{dt} = t + y$ ,  $y(0) = 1$  in the interval  $[0, 0.4]$  with  $h = 0.1$ , using Milne's P-C method.

### Solution

Milne's P-C method demand the solution at first four points  $t_0, t_1, t_2$  and  $t_3$ . As it is not a self – starting method, we shall use R-K method of fourth order to get the required solution and then switch over to Milne's P – C method.

Thus, taking  $t_0 = 0, t_1 = 0.1, t_2 = 0.2, t_3 = 0.3$  we get the corresponding  $y$  values using

<sup>th</sup>  
R–K method of 4 order; that is  $y_0 = 1, y_1 = 1.1103, y_2 = 1.2428$  and  $y_3 = 1.3997$   
(Reference Lecture 38)

Now we compute

$$y_1' = t_1 + y_1 = 0.1 + 1.1103 = 1.2103$$

$$y_2' = t_2 + y_2 = 0.2 + 1.2428 = 1.4428$$

$$y_3' = t_3 + y_3 = 0.3 + 1.3997 = 1.6997$$

Using Milne's predictor formula

$$P: y_4 = y_0 + \frac{4h}{3} (2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.5)}{3} [2(1.21103) - 1.4428 + 2(1.69971)]$$

$$= 1.58363$$

Before using corrector formula, we compute

$$y_4' = t_4 + y_4(\text{predicted value})$$

$$= 0.4 + 1.5836 = 1.9836$$

Finally, using Milne's corrector formula, we compute

$$\begin{aligned}
 C: y_4 &= y_2 + \frac{h}{3}(y'_4 + 4y'_3 + y'_2) \\
 &= 1.2428 + \frac{0.1}{3}(1.9836 + 6.7988 + 1.4428) \\
 &= 1.5836
 \end{aligned}$$

The required solution is:

|   |   |        |        |        |        |
|---|---|--------|--------|--------|--------|
| t | 0 | 0.1    | 0.2    | 0.3    | 0.4    |
| y | 1 | 1.1103 | 1.2428 | 1.3997 | 1.5836 |

### Example

Using Milne's Predictor-Corrector Formula find  $f(0.4)$  from Ordinary Differential Equation

$$y' = x - y ; y(0) = 1 ; h = 0.1$$

with the help of following table.

|   |   |        |        |        |
|---|---|--------|--------|--------|
| X | 0 | 0.1    | 0.2    | 0.3    |
| Y | 1 | 0.9097 | 0.8375 | 0.7816 |

### Solution:

Here,

$$x_0 = 0, \quad x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3, \quad x_4 = 0.4$$

$$y_1' = x_1 - y_1 = 0.1 - 0.9097 = -0.8097$$

$$y_2' = x_2 - y_2 = 0.2 - 0.8375 = -0.6375$$

$$y_3' = x_3 - y_3 = 0.3 - 0.7816 = -0.4816$$

Now, using Predictor Formula

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$y_4 = 1 + \frac{4 \cdot 0.1}{3}(-1.9451)$$

$$y_4 = 0.740653333$$

Using the predicted value, we shall now compute the corrected value as;

$$y_4 = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

Now,

$$y_4' = x_4 - y_4 = 0.4 - 0.740653333 = -0.340653333$$

Putting the values, into the Corrector Formula;

$$y_4 = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$y_4 = 0.8375 + \frac{0.1}{3}(-0.6375 + (4 \times -0.4816) - 0.340653333)$$

$$y_4 = 0.8375 - 0.096818444$$

$$y_4 = 0.740681556 \quad \text{Ans.}$$



### Adam-Moulton Method

It is another predictor-corrector method, where we use the fact that the solution to the given initial value problem is known at past four equally spaced points

$$t_n, t_{n-1}, t_{n-2}, t_{n-3}.$$

The task is to compute the value of  $y$  at  $t_{n+1}$

Let us consider the differential equation

$$\frac{dy}{dt} = f(t, y)$$

Integrating between the limits  $t_n$  to  $t_{n+1}$ , we have

$$\int_{t_n}^{t_{n+1}} \frac{dy}{dt} dt = \int_{t_n}^{t_{n+1}} f(t, y) dt$$

That is

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} f(t, y) dt$$

To carry out integration, we proceed as follows. We employ *Newton's backward interpolation formula*, so that

$$f(t, y) = f_n + s \nabla f_n + \frac{s(s+1)}{2} \nabla^2 f_n + \frac{s(s+1)(s+2)}{6} \nabla^3 f_n + \dots$$

Where

$$s = \frac{t - t_n}{h}$$

After substitution, we obtain

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} \left[ f_n + s \nabla f_n + \frac{s(s+1)}{2} \nabla^2 f_n + \frac{s(s+1)(s+2)}{6} \nabla^3 f_n + \frac{s(s+1)(s+2)(s+3)}{24} \nabla^4 f_n + \dots \right] dt$$

Now by changing the variable of integration (from  $t$  to  $s$ ), the limits of integration also changes (from 0 to 1), and thus the above expression becomes

$$y_{n+1} - y_n = h \int_0^1 \left[ f_n + s \nabla f_n + \frac{s(s+1)}{2} \nabla^2 f_n + \frac{s(s+1)(s+2)}{6} \nabla^3 f_n + \frac{s(s+1)(s+2)(s+3)}{24} \nabla^4 f_n + \dots \right] ds$$

Actual integration reduces the above expression to

$$y_{n+1} - y_n = h \left( f_n + \frac{1}{2} \nabla f_n + \frac{5}{12} \nabla^2 f_n + \frac{3}{8} \nabla^3 f_n + \frac{251}{720} \nabla^4 f_n \right)$$

Now substituting the differences such as

$$\nabla f_n = f_n - f_{n-1}$$

$$\begin{aligned}\nabla^2 f_n &= f_n - 2f_{n-1} + f_{n-2} \\ \nabla^3 f_n &= f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}\end{aligned}$$

Equation simplifies to

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) + \frac{251}{720}h\nabla^4 f_n$$

Alternatively, it can be written as

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}] + \frac{251}{720}h\nabla^4 y'_n$$

This is known as Adam's predictor formula.

The truncation error is  $(251/720)h\nabla^4 y'_n$ .

To obtain corrector formula, we use Newton's backward interpolation formula about  $f_{n+1}$  instead of  $f_n$ .

We obtain

$$\begin{aligned}y_{n+1} = y_n + h \int_{-1}^0 \left[ f_{n+1} + s\nabla f_{n+1} + \frac{s(s+1)}{2} \nabla^2 f_{n+1} \right. \\ \left. + \frac{s(s+1)(s+2)}{6} \nabla^3 f_{n+1} + \frac{s(s+1)(s+2)(s+3)}{24} \nabla^4 f_{n+1} + \dots \right] ds\end{aligned}$$

Carrying out the integration and repeating the steps, we get the corrector formula as

$$y_{n+1} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}) + \left(\frac{-19}{720}\right)h\nabla^4 y'_{n+1}$$

Here, the truncation error is  $(19/20)h\nabla^4 y'_{n+1}$ .

The truncation error in Adam's predictor is approximately thirteen times more than that in the corrector, but with opposite sign.

In general, Adam-Moulton predictor-corrector pair can be written as

$$\begin{aligned}P: y_{n+1} &= y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}) \\ C: y_{n+1} &= y_n + \frac{h}{24}(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2})\end{aligned}$$

### Example

Using Adam-Moulton predictor-corrector method, find the solution of the initial value problem

$$\frac{dy}{dt} = y - t^2, \quad y(0) = 1$$

at  $t = 1.0$ , taking  $h = 0.2$ . Compare it with the analytical solution.

### Solution

In order to use Adam's P-C method, we require the solution of the given differential equation at the past four equally spaced points, for which we use R-K method of 4<sup>th</sup> order which is self starting.

Thus taking  $t_0 = 0, y_0 = 1, h = 0.2$ , we compute

$$\begin{aligned} k_1 &= 0.2, k_2 = 0.218, \\ k_3 &= 0.2198, k_4 = 0.23596, \end{aligned}$$

and get

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.21859$$

Taking  $t_1 = 0.2$ ,

$y_1 = 1.21859, h = 0.2$ ,  
we compute  $k_1 = 0.23571$ ,

$$k_2 = 0.2492,$$

$k_3 = 0.25064, k_4 = 0.26184$ , and get

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.46813$$

Now, we take  $t_2 = 0.4$ ,

$y_2 = 1.46813, h = 0.2$ , and compute  $k_1 = 0.2616$ ,

$$k_2 = 0.2697,$$

$$k_3 = 0.2706, k_4 = 0.2757$$

to get

$$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.73779$$

Thus, we have at our disposal

$$y_0 = y(0) = 1$$

$$y_1 = y(0.2) = 1.21859$$

$$y_2 = y(0.4) = 1.46813$$

$$y_3 = y(0.6) = 1.73779$$

Now, we use Adam's P-C pair to calculate  $y(0.8)$  and  $y(1.0)$  as follows:

$$P: y_{n+1} = y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$$

$$C: y_{n+1} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2})$$

$$\text{Thus } y_4^P = y_3 + \frac{h}{24}(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0) \quad (1)$$

From the given differential equation, we have

$$y' = y - t^2.$$

Therefore

$$\begin{aligned}
 y'_0 &= y_0 - t_0^2 = 1.0 \\
 y'_1 &= y_1 - t_1^2 = 1.17859 \\
 y'_2 &= y_2 - t_2^2 = 1.30813 \\
 y'_3 &= y_3 - t_3^2 = 1.37779
 \end{aligned}$$

Hence, from Eq. (1), we get

$$y(0.8) = y_4^p = 1.73779 + \frac{0.2}{24}(75.77845 - 77.17967 + 43.60783 - 9) = 2.01451$$

Now to obtain the corrector value of  $y$  at  $t = 0.8$ , we use

$$y_4^c = y^c(0.8) = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1') \quad (2)$$

But  $9y_4' = 9(y_4^p - t_4^2) = 9[2.01451 - (0.8)^2] = 12.37059$

Therefore

$$y_4^c = y^c(0.8) = 1.73779 + \frac{0.2}{24}(12.37059 + 26.17801 - 6.54065 + 1.17859) = 2.01434 \quad (3)$$

Proceeding similarly, we get

$$y_5^p = y^p(1.0) = y_4 + \frac{h}{24}(55y_4' - 59y_3' + 37y_2' - 9y_1')$$

Noting that

$$y_4' = y_4 - t_4^2 = 1.3743,$$

We calculate

$$y_5^p = 2.01434 + \frac{0.2}{24}(75.5887 - 81.28961 + 48.40081 - 10.60731) = 2.28178$$

Now, the corrector formula for computing  $y_5$  is given by

$$y_5^c = y^c(1.0) = y_4 + \frac{h}{24}(9y_5' + 19y_4' - 5y_3' + y_2') \quad (4)$$

But

$$9y_5' = 9(y_5^p - t_5^2) = 11.53602$$

Then finally we get

$$\begin{aligned}
 y_5^c &= y(1.0) = 2.01434 + \frac{0.2}{24}(11.53602 + 26.17801 - 6.54065 + 1.17859) \\
 &= 2.28393
 \end{aligned} \quad (5)$$

The analytical solution can be seen in the following steps.

$$\frac{dy}{dt} - y = -t^2$$

After finding integrating factor and solving, we get

$$\frac{d}{dt} ye^{-t} = -e^{-t}t^2$$

Integrating, we get

$$ye^{-t} = -\int e^{-t}t^2 dt = \int t^2 d(e^{-t}) = t^2 e^{-t} + 2te^{-t} + c$$

That is

$$y = t^2 + 2t + 2 + \frac{c}{e^{-t}}$$

Now using the initial condition,  $y(0) = 1$ ,

we get  $c = -1$ .

Therefore, the analytical solution is given by

$$y = t^2 + 2t + 2 - e^t$$

From which we get

$$y(1.0) = 5 - e = 2.2817$$

### Example

Using Adam-Moulton Predictor-Corrector Formula find  $f(0.4)$  from Ordinary Differential Equation

$$y' = 1 + 2xy \quad ; \quad y(0) = 0 \quad ; \quad h = 0.1$$

with the help of following table.

|   |   |        |        |        |
|---|---|--------|--------|--------|
| X | 0 | 0.1    | 0.2    | 0.3    |
| Y | 0 | 0.1007 | 0.2056 | 0.3199 |

### Solution:

Here

$$h = 0.1$$

$$f(x, y) = 1 + 2xy$$

$$y_0' = 1 + 2x_0 y_0 = 1 + 2(0)(0) = 1$$

$$y_1' = 1 + 2x_1 y_1 = 1 + 2(0.1)(0.1007) = 1.02014$$

$$y_2' = 1 + 2x_2 y_2 = 1 + 2(0.2)(0.2056) = 1.08224$$

$$y_3' = 1 + 2x_3 y_3 = 1 + 2(0.3)(0.3199) = 1.19194$$

Now, Using Adam's P-C Pair Formula:-

$$y_{n+1} = y_n + \frac{h}{24} (55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$$

*Putting the values;*

$$y_4 = y_3 + \frac{h}{24} (55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$$

$$y_4 = 0.3199 + \frac{0.1}{24} (55(1.19194) - 59(1.08224) + 37(1.02014) - 9(1))$$

$$y_4 = 0.446773833$$

Computing  $y'_4$  for the Corrector Formula;

$$y'_4 = 1 + 2x_4 y_4 = 1 + 2(0.4)(0.446773833)$$

$$y'_4 = 1.3574190664$$

Now Applying the Corrector Formula;

$$y_{n+1} = y_n + \frac{h}{24} (9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2})$$

$$y_4 = y_3 + \frac{h}{24} (9y'_4 + 19y'_3 - 5y'_2 + y'_1)$$

$$y_4 = 0.3199 + \frac{0.1}{24} (9(1.3574190664) + 19(1.19194) - 5(1.08224) + 1.02014)$$

$$y_4 = 0.446869048$$

### Convergence and Stability Considerations

The numerical solution of a differential equation can be shown to converge to its exact solution, if the step size  $h$  is very small. The numerical solution of a differential equation is said to be stable if the error do not grow exponentially as we compute from one step to another. Stability consideration are very important in finding the numerical solutions of the differential equations either by *single-step* methods or by using *multi-step* methods. However, theoretical analysis of stability and convergence of R -K methods and P –C

methods are highly involved and obtain numerically stable solution using 4<sup>th</sup> order R – K method to the simple problem  $y' = Ay$  gives us stability condition as  $-2.78 \leq Ah$

In practice, to get numerically stable solutions to similar problems, we choose the value of  $h$  much smaller than the value given by the above condition and also check for consistency of the result.

Another topic of interest which is not considered, namely the stiff system of differential equations that arises in many chemical engineering systems, such as chemical reactors, where the rate constants for the reactions involved are widely different.

Most of the realistic stiff DE do not have analytical solutions and therefore only numerical solutions can be obtained. However, to get numerically stable solutions, a very small step size  $h$  is required, to use either R-K methods or P – C methods.

More computer time is required

## Lecture # 41

### Examples of Differential Equations

#### Recall EULER METHOD

We considered the differential equation of first order with the initial condition  $y(t_0) = y_0$ .

$$\frac{dy}{dt} = f(t, y)$$

We obtained the solution of the given differential equation in the form of a recurrence relation

$$y_{m+1} = y_m + hf(t_m, y_m)$$

In fact Euler's method constructs  $w_i \sim y(t_i)$  for each  $i = 0, 1, \dots, N-1$  by deleting the remainder term. Thus the Euler's Method is

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

for each  $i = 0, 1, \dots, N-1$

Euler's algorithm

Let us try to approximate the solution of the given IVP at  $(N+1)$  equally spaced numbers in the interval  $[a, b]$

$$y' = f(t, y),$$

$$a \leq t \leq b, \quad y(a) = \alpha$$

INPUT endpoints  $a, b$ ; integer  $N$ , initial condition ( $\alpha$ )

OUTPUT approximate  $w$  to  $y$  at the  $(N+1)$  values of  $t$

Step 1

Set  $h = (b-a) / N$

$$t = a$$

$$w = (\alpha)$$

OUTPUT  $(t, w)$

Step 2

For  $i = 0, 1, \dots, N$  do Step 3, 4.

Step 3

Set  $w = w + hf(t, w)$ ; (compute  $w_i$ ).

$$t = a + i h \text{ (compute } t_i \text{)}$$

Step 4 OUTPUT  $(t, w)$

Step 5 STOP

Example

Use Euler's method to approximate the solution of IVP

$$y' = y - t^2 + 1, \quad 0 < t < 2,$$

$$y(0) = 0.5 \text{ with } N = 10.$$

Solution

Here,  $h = 0.2$ ,  $t_i = 0.2i$ ,  $w_0 = 0.5$  and  $w_{i+1} = w_i + h(w_i - t_i^2 + 1)$

$$= w_i + 0.2[w_i - 0.04i^2 + 1]$$

$$= 1.2 w_i - 0.008i^2 + 0.2$$

$$\text{for } i = 0, 1, \dots, 9.$$

The exact solution is

$$y(t) = (t+1)^2 - 0.5 e^t$$

> alg051();

This is Euler's Method.

Input the function  $F(t, y)$  in terms of  $t$  and  $y$

For example:  $y=t^2+1$   
 >  $y=t^2+1$   
 Input left and right endpoints separated by blank  
 > 0 2  
 Input the initial condition  
 > 0.5  
 Input a positive integer for the number of subintervals  
 > 10

Choice of output method:

1. Output to screen
2. Output to text file

Please enter 1 or 2

> 1

Output

| t     | w         |
|-------|-----------|
| 0.000 | 0.5000000 |
| 0.200 | 0.8000000 |
| 0.400 | 1.1520000 |
| 0.600 | 1.5504000 |
| 0.800 | 1.9884800 |
| 1.000 | 2.4581760 |

> alg051();

This is Euler's Method.

Input the function F (t,y) in terms of t and y

For example:  $y-3*t^2+4$

>  $y-3*t^2+4$

Input left and right hand points separated by a blank

> 0 1

Input the initial condition

> 0.5

Input a positive integer for the number of subintervals

> 10

Choice of output method:

1. Output to screen
2. Output to text file

Please enter 1 or 2

> 1

Output

| t     | w         |
|-------|-----------|
| 0.000 | 0.5000000 |
| 0.100 | 0.9500000 |
| 0.200 | 1.4420000 |
| 0.300 | 1.9742000 |
| 0.400 | 2.5446200 |
| 0.500 | 3.1510820 |
| 0.600 | 3.7911902 |
| 0.700 | 4.4623092 |
| 0.800 | 5.1615401 |
| 0.900 | 5.8856942 |
| 1.000 | 6.6312636 |

Recall Runge-Kutta (Order Four) METHOD

The fourth-order R-K method was described as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}
k_1 &= hf(t_n, y_n) \\
k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 &= hf(t_n + h, y_n + k_3)
\end{aligned}$$

Example

Solve the following differential equation

$\frac{dy}{dt} = t + y$  with the initial condition  $y(0) = 1$ , using fourth- order Runge-Kutta method from  $t = 0$  to  $t = 0.4$  taking  $h = 0.1$

Solution

The fourth-order Runge-Kutta method is described as

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \dots \dots \dots (1)$$

where

$$\begin{aligned}
k_1 &= hf(t_n, y_n) \\
k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 &= hf(t_n + h, y_n + k_3)
\end{aligned}$$

In this problem,

$$f(t, y) = t + y, \quad h = 0.1, \quad t_0 = 0, \quad y_0 = 1.$$

As a first step, we calculate

$$k_1 = hf(t_0, y_0) = 0.1(1) = 0.1$$

$$\begin{aligned}
k_2 &= hf(t_0 + 0.05, y_0 + 0.05) \\
&= hf(0.05, 1.05) = 0.1[0.05 + 1.05] \\
&= 0.11
\end{aligned}$$

$$\begin{aligned}
k_3 &= hf(t_0 + 0.05, y_0 + 0.055) \\
&= 0.1(0.05 + 1.055) \\
&= 0.1105
\end{aligned}$$

$$k_4 = 0.1(0.1 + 1.1105) = 0.12105$$

Now, we compute from

$$\begin{aligned}
y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) \\
&= 1.11034
\end{aligned}$$

Therefore  $y(0.1) = y_1 = 1.1103$

In the second step, we have to find  $y_2 = y(0.2)$

We compute

$$k_1 = hf(t_1, y_1) = 0.1(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1[0.15 + (1.11034 + 0.060517)] = 0.13208$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1[0.15 + (1.11034 + 0.06604)] = 0.132638$$

$$k_4 = hf(t_1 + h, y_1 + k_3)$$

$$= 0.1[0.2 + (1.11034 + 0.132638)] = 0.1442978$$

From Equation (1), we see that

$$y_2 = 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208)$$

$$+ 2(0.132638) + 0.1442978] = 1.2428$$

Similarly we calculate,

$$k_1 = hf(t_2, y_2) = 0.1[0.2 + 1.2428] = 0.14428$$

$$k_2 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1[0.25 + (1.2428 + 0.07214)] = 0.156494$$

$$k_3 = hf\left(t_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1[0.3 + (1.2428 + 0.078247)] = 0.1571047$$

$$k_4 = hf(t_2 + h, y_2 + k_3) = 0.1[0.3 + (1.2428 + 0.1571047)] = 0.16999047$$

Using equation (1), we compute

$$y(0.3) = y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.399711$$

Finally, we calculate

$$k_1 = hf(t_3, y_3) = 0.1[0.3 + 1.3997] = 0.16997$$

$$k_2 = hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1[0.35 + (1.3997 + 0.084985)] = 0.1834685$$

$$k_3 = hf\left(t_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1[0.35 + (1.3997 + 0.091734)] = 0.1841434$$

$$k_4 = hf(t_3 + h, y_3 + k_3) = 0.1[0.4 + (1.3997 + 0.1841434)] = 0.19838434$$

Using them in equation (1), we get

$$y(0.4) = y_4$$

$$= y_3 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.58363$$

which is the required result  
Runge-Kutta Order Four

$$w_0 = \alpha$$

$$k_1 = hf(t_i, w_i)$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_i + h, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \dots\dots\dots(1)$$

RK4 algorithm

Let us try to approximate the solution of the given IVP at (N+1) equally spaced numbers in the interval [a ,b]

$$y' = f(t, y),$$

$$a \leq t \leq b, \quad y(a) = \alpha$$

INPUT endpoints a, b; integer N, initial condition (alpha)

OUTPUT approximate w to y at the (N+1) values of t

Step 1

Set  $h = (b-a) / N$

$$t = a$$

$$w = (\text{alpha})$$

OUTPUT (t , w)

Step 2

For i = 0, 1, ... N do Step 3 - 5.

Step 3

Set

$$K_1 = hf(t, w)$$

$$K_2 = hf\left(t + \frac{h}{2}, w + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(t + \frac{h}{2}, w + \frac{K_2}{2}\right)$$

$$K_4 = hf(t + h, w + K_3)$$

$$w_{i+1} = w_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

Step 3

Set  $w = w + h f(t, w)$ ; (compute  $w_i$ ).

$$t = a + i h \text{ (compute } t_i \text{)}$$

Step 4 OUTPUT (t , w)

```

Step 5 STOP
> alg052();
This is the Runge-Kutta Order Four Method.
Input the function F(t,y) in terms of t and y
For example: y-t^2+1
> y-t^2+1
Input left and right endpoints separated by blank
> 0 2
Input the initial condition
> 0.5
Input a positive integer for the number of subintervals
> 10
Choice of output method:
1. Output to screen
2. Output to text file
Please enter 1 or 2
> 1
Output
  t      w
0.000 0.5000000
0.200 0.8292933
0.400 1.2140762
0.600 1.6489220
0.800 2.1272027
1.000 2.6408227
1.200 3.1798942
1.400 3.7323401
1.600 4.2834095
1.800 4.8150857
2.000 5.3053630

```

## Examples of Numerical Differentiation

The simplest formula for differentiation is

$$f'(x_0) = \frac{1}{h} [f(x_0 + h) - f(x_0)] - \frac{h^2}{2} f^{(2)}(\xi),$$

## Example

Let  $f(x) = \ln x$  and  $x_0 = 1.8$ . Then quotient

$$\frac{f(1.8 + h) - f(1.8)}{h}, \quad h > 0,$$

is used to approximate  $f'(1.8)$  with error

$$\frac{|hf''(\xi)|}{2} = \frac{|h|}{2\xi^2} \leq \frac{|h|}{2(1.8)^2}, \quad \text{where } 1.8 < \xi < 1.8 + h.$$

Let us see the results for  $h = 0.1, 0.01$ , and  $0.001$ .

| $h$   | $f(1.8 + h)$ | $\frac{f(1.8 + h) - f(1.8)}{h}$ | $\frac{ h }{2(1.8)^2}$ |
|-------|--------------|---------------------------------|------------------------|
| 0.1   | 0.64185389   | 0.5406722                       | 0.0154321              |
| 0.01  | 0.59332685   | 0.5540180                       | 0.0015432              |
| 0.001 | 0.58834207   | 0.5554013                       | 0.0001543              |

Since  $f'(x) = 1/x$ ,

The exact value of  $f'(1.8)$  is  $0.555\bar{5}$  and the error bounds are appropriate.

The following two three point formulas become especially useful if the nodes are equally spaced, that is, when

$$x_1 = x_0 + h \text{ and}$$

$$x_2 = x_0 + 2h,$$

$$f'(x_0) = \frac{1}{2h} [-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + \frac{h^2}{3} f^{(3)}(\xi_0),$$

where  $\xi_0$  lies between  $x_0$  and  $x_0 + 2h$ , and

$$f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f^{(3)}(\xi_1),$$

where  $\xi_1$  lies between  $(x_0 - h)$  and  $(x_0 + h)$ .

Given in Table below are values for  $f(x) = xe^x$ .

| x   | f (x)     |
|-----|-----------|
| 1.8 | 10.889365 |
| 1.9 | 12.703199 |
| 2.0 | 14.778112 |
| 2.1 | 17.148957 |
| 2.2 | 19.855030 |

Since

$$f'(x) = (x+1)e^x, f'(2.0) = 22.167168.$$

Approximating  $f'(2.0)$

using the various three-and five-point formulas produces the following results.

**Three point formulas:**

$$f'(x_0) = \frac{1}{2h} [-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + \frac{h^2}{3} f^{(3)}(\xi_0),$$

$$f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f^{(3)}(\xi_1),$$

Using three point formulas we get

$$h = 0.1: \frac{1}{0.2} [-3f(2.0) + 4f(2.1) - f(2.2)]$$

$$= 22.032310,$$

$$h = -0.1: \frac{1}{-0.2} [-3f(2.0) + 4f(1.9) - f(1.8)]$$

$$= 22.0054525,$$

$$h = 0.1: \frac{1}{0.2} [f(2.1) - f(1.9)]$$

$$= 22.228790,$$

$$h = 0.2: \frac{1}{0.4} [f(2.2) - f(1.8)]$$

$$= 22.414163.$$

#### Five point formula

Using the five point formula with  $h = 0.1$  (the only formula applicable):

$$f'(x_0) = \frac{1}{12h} [f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)]$$

$$= \frac{1}{0.2} [f(1.8) - 8f(1.9) + 8f(2.1) - f(2.2)]$$

$$= 22.166999.$$

The errors in the formulas are approximately

$$1.35 \times 10^{-1}, 1.13 \times 10^{-1}, -6.16 \times 10^{-2}, -2.47 \times 10^{-1},$$

$$\text{and } 1.69 \times 10^{-4},$$

respectively. Clearly, the five-point formula gives the superior result.

Consider approximating for  $f(x) = \sin x$ , using the values in table [ the true value is  $\cos(0.900) = 0.62161$ .]

| x     | sin x   | x     | sin x   |
|-------|---------|-------|---------|
| 0.800 | 0.71736 | 0.901 | 0.78395 |
| 0.850 | 0.75128 | 0.902 | 0.78457 |
| 0.880 | 0.77074 | 0.905 | 0.78643 |
| 0.890 | 0.77707 | 0.910 | 0.78950 |
| 0.895 | 0.78021 | 0.920 | 0.79560 |
| 0.898 | 0.78208 | 0.950 | 0.81342 |
| 0.899 | 0.78270 | 1.000 | 0.84147 |

Using the formula

$$f'(0.900) \approx \frac{f(0.900 + h) - f(0.900 - h)}{2h}$$

with different values of  $h$  gives the approximations in table given below:

| $h$   | Approximation to $f'(0.900)$ | Error    |
|-------|------------------------------|----------|
| 0.001 | 0.62500                      | 0.00339  |
| 0.002 | 0.62250                      | 0.00089  |
| 0.005 | 0.62200                      | 0.00039  |
| 0.010 | 0.62150                      | -0.00011 |
| 0.020 | 0.62150                      | -0.00011 |
| 0.050 | 0.62140                      | -0.00021 |
| 0.100 | 0.62055                      | -0.00106 |

## Examples of Numerical Integration

### EXAMPLE

The Trapezoidal rule for a function  $f$  on the interval  $[0, 2]$  is

$$\int_0^2 f(x) dx \approx \frac{2}{2} [f(0) + f(2)],$$

$$\int_0^2 f(x) dx \approx f(0) + f(2),$$

while Simpson's rule for  $f$  on  $[0, 2]$  is

$$\int_0^2 f(x) dx \approx \frac{2}{3} [f(0) + 4f(1) + f(2)].$$

That is

$$\int_0^2 f(x) dx \approx \frac{1}{3} [f(0) + 4f(1) + f(2)].$$

$$f(x) \quad x^2 \quad x^4 \quad 1/(x+1) \quad \sqrt{1+x^2} \quad \sin x \quad ex$$

Exact value    2.667    6.400    1.099    2.958    1.416    6.389

Trapezoidal    4.000    16.000    1.333    3.326    0.909    8.389

Simpson's    2.667    6.667    1.111    2.964    1.425    6.421

Use close and open formulas listed below to approximate

$$\int_0^{\pi/4} \sin x dx = 1 - \sqrt{2}/2$$

Some of the common closed Newton-Cotes formulas with their error terms are as follows:

$n = 1$ : Trapezoidal rule

$$\int_{x_0}^{x_1} f(x) dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f'(\xi),$$

Where  $x_0 < \xi < x_1$ .

$n = 2$ : Simpson's rule

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{2} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi), \quad x_0 < \xi < x_2.$$

n = 3: Simpson's rule

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8} [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(\xi), \quad x_0 < \xi < x_3.$$

n = 4:

$$\int_{x_2}^{x_4} f(x)dx = \frac{2h}{45} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] - \frac{8h^7}{945} f^{(6)}(\xi)$$

Where  $x_0 < \xi < x_4$ .

n = 0: Midpoint rule

$$\int_{x_2}^{x_1} f(x)dx = 2hf(x_0) + \frac{h^3}{3} f'(\xi), \quad \text{where } x_{-1} < \xi < x_1.$$

| n               | 1          | 2          | 3          | 4          |
|-----------------|------------|------------|------------|------------|
| Closed formulas | 0.27768018 | 0.29293264 | 0.29291070 | 0.29289318 |
| Error           | 0.01521303 | 0.00003942 | 0.00001748 | 0.00000004 |
| Open formulas   | 0.29798754 | 0.29285866 | 0.29286923 |            |
| Error           | 0.00509432 | 0.00003456 | 0.00002399 |            |

## Composite Numerical Integration

### EXAMPLE 1

Consider approximating  $\int_0^\pi \sin x dx$  with an absolute error less than 0.00002, using the

Composite Simpson's rule. The Composite Simpson's rule gives

$$\int_0^\pi \sin x dx = \frac{h}{3} \left[ 2 \sum_{j=1}^{(n/2)-1} \sin x_{2j} + 4 \sum_{j=1}^{n/2} \sin x_{2j-1} \right] - \frac{\pi h^4}{180} \sin \mu.$$

Since the absolute error is required to be less than 0.00002, the inequality

$$\left| \frac{\pi h^4}{180} \sin \mu \right| \leq \frac{\pi h^4}{180} = \frac{\pi^5}{180n^4} < 0.00002$$

is used to determine n and h. Computing these calculations gives n greater than or equal to 18. If

$$n = 20, \text{ then the formula becomes } \int_0^\pi \sin x dx \approx \frac{\pi}{60} \left[ 2 \sum_{j=1}^9 \sin \left( \frac{j\pi}{10} \right) + 4 \sum_{j=1}^{10} \sin \left( \frac{(2j-1)\pi}{20} \right) \right] = 2.000006.$$

To be assured of this degree of accuracy using the Composite Trapezoidal rule requires that

$$\left| \frac{\pi h^2}{12} \sin \mu \right| \leq \frac{\pi h^2}{12} = \frac{\pi^3}{12n^2} < 0.00002$$

or that  $n \geq 360$ . Since this is many more

calculations than are needed for the

Composite Simpson's rule, it is clear

that it would be undesirable to use the Composite Trapezoidal rule on this problem. For

comparison purposes, the Composite Trapezoidal rule with  $n = 20$  and

$$\begin{aligned} \int_0^\pi \sin x dx &\approx \frac{\pi}{20} \left[ \sin 0 + \sin \pi \right] \\ &= \frac{\pi}{20} \left[ 2 \sum_{j=1}^{19} \sin \left( \frac{j\pi}{20} \right) + \sin 0 + \sin \pi \right] \\ &= \frac{\pi}{20} \left[ 2 \sum_{j=1}^{19} \sin \left( \frac{j\pi}{20} \right) \right] = 1.9958860. \end{aligned}$$

The exact answer is 2; so Simpson's rule with  $n = 20$  gave an answer well within the required error bound, whereas the Trapezoidal rule with  $n = 20$  clearly did not.

**An Example of Industrial applications:** A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy$$

a) use single segment Trapezoidal rule to find the probability that there are 250 or more sheets.

b) Find the true error, Et for part (a).

C) Find the absolute relative true error for part (a).

$$I \approx (b-a) \left[ \frac{f(a) + f(b)}{2} \right] \text{ where}$$

$$a = 250 \quad b = 270$$

$$f(y) = 0.3515e^{-0.3881(y-252.2)^2} \quad f(250) = 0.3515e^{-0.3881(250-252.2)^2} = 0.053721$$

$$f(270) = 0.3515e^{-0.3881(270-252.2)^2} = 1.3888 \times 10^{-54}$$

$$I = (270 - 250) \left[ \frac{0.053721 + 1.3888 \times 10^{-54}}{2} \right]$$

$$= 0.53721$$

b) The exact value of the above integral cannot be found. We assume the value obtained by adaptive numerical integration using Maple as the exact value for calculating the true error and relative true error.

$$P(y \geq 250) = \int_{250}^{270} 0.3515 e^{-0.3881(y-252.2)^2} dy = 0.97377$$

$$\text{so the true error is } = 0.97377 - 0.53721 = 0.43656$$

The absolute relative true error,

$|\epsilon_t|$ , would then be

$$|\epsilon_t| = \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100$$

$$= \left| \frac{0.97377 - 0.53721}{0.97377} \right| \times 100$$

$$= 44.832\%$$

## Improper Integrals

### EXAMPLE

To approximate the values of the improper integral

$$\int_0^1 \frac{e^x}{\sqrt{x}} dx,$$

we will use the Composite Simpson's rule with  $h = 0.25$ . Since the fourth Taylor polynomial for  $e^x$  about  $x = 0$  is

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

We have

$$\int_0^1 \frac{P_4(x)}{\sqrt{x}} dx = \lim_{M \rightarrow 0^+} \left[ \frac{2x^{1/2}}{3} + \frac{2}{5}x^{3/2} + \frac{1}{21}x^{5/2} + \frac{1}{21}x^{7/2} + \frac{1}{180}x^{9/2} \right]_M^1$$

$$= 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{108} \approx 2.9235450.$$

Table below lists the approximate values of

$$G(x) = \begin{cases} \frac{e^x - P_4(x)}{\sqrt{4}} & \text{when } 0 < x \leq 1 \\ 0, & \text{when } x = 0 \end{cases}$$

| x    | G(x)      |
|------|-----------|
| 0.00 | 0         |
| 0.25 | 0.0000170 |
| 0.50 | 0.0004013 |
| 0.75 | 0.0026026 |
| 1.00 | 0.0099485 |

Applying the Composite Simpson's rule to G using these data gives

$$\begin{aligned} \int_0^1 G(x) dx &\approx \\ \frac{0.25}{3} [0 + 4(0.0000170) + 2(0.0004013) \\ &+ 4(0.0026026) + 0.0099485] \\ &= 0.0017691 \end{aligned}$$

Hence

$$\int_0^1 \frac{e^x}{\sqrt{x}} dx \approx 2.9235450 + 0.0017691 = 2.9253141.$$

This result is accurate within the accuracy of the Composite Simpson's rule approximation for the function G. Since  $|G^{(4)}(x)| < 1$

on  $[0, 1]$ , the error is bounded by

$$\frac{1-0}{180} (0.25)^4 (1) = 0.0000217.$$

### EXAMPLE

To approximate the value of the improper integral

$$I = \int_1^{\infty} x^{-3/2} \sin \frac{1}{x} dx.$$

we make the change of variable  $t = x^{-1}$  to obtain  $I = \int_0^1 t^{1/2} \sin t \, dt$ .

The fourth Taylor polynomial,  $P_4(t)$ , for  $\sin t$  about 0 is

$$P_4(t) = t - \frac{1}{6}t^3,$$

$$\begin{aligned} \text{So we have } I &= \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}} dt + \int_0^1 \frac{t^{1/2}}{6} dt \\ &= \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}} dt + \left[ \frac{2}{3}t^{3/2} - \frac{1}{21}t^{7/2} \right]_0^1 \\ &= \int_0^1 \frac{\sin t - t + \frac{1}{6}t^3}{t^{1/2}} dt + 0.61904761. \end{aligned}$$

Applying the Composite Simpson's rule with  $n = 8$  to the remaining integral gives  $I = 0.0014890097 + 0.61904761 = 0.62053661$ ,

which is accurate to within  $4.0 \times 10^{-8}$ .

## An Introduction to MAPLE

Maple is a comprehensive computer system for advanced mathematics.

It includes facilities for interactive algebra, calculus, discrete mathematics, graphics, numerical computation etc.

It provides a unique environment for rapid development of mathematical programs using its vast library of built-in functions and operations.

Syntax :As with any computer language, Maple has its own syntax.

We try to explain some of the symbols used in Maple

| Symbol   | Description                                                       | Examples                     | Sample Output              |
|----------|-------------------------------------------------------------------|------------------------------|----------------------------|
| ;        | End-of-line. Tells Maple to process the line and show the output. | hello;                       | hello                      |
| :        | End-of-line. Tells Maple to process the line and hide the output. | hello:                       |                            |
| :=       | Assignment. Lets you assign values to variables.                  | a := 3;<br>a;                | a := 3<br>3                |
| +, -     | Addition, subtraction.                                            | 1 + 3;<br>1 - 3;             | 4<br>-2                    |
| *, /     | Multiplication, division                                          | 3*412;<br>1236/3;<br>7/3;    | 1236<br>412<br>7/3         |
| ^, sqrt  | Power, square root                                                | 2^3;<br>sqrt(2);<br>2^(1/2); |                            |
| evalf, . | Floating-point (decimal) evaluation                               | evalf(7/3);<br>7.0/3;        | 2.333333333<br>2.333333333 |

|       |                                                                |                                    |                          |
|-------|----------------------------------------------------------------|------------------------------------|--------------------------|
| I, Pi | Imaginary unit, Pi.                                            | 2 + 3*I;<br>(2*I)^2;<br>evalf(Pi); | 2+3I<br>-4<br>3.14159265 |
| %, %% | Recall the last output, recall the second-to-last output, etc. | %;<br>%%%;                         | 3.14159265-4             |

Some syntactical Tips:

Maple is case sensitive. foo, Foo, and FOO are three different things.

x\*y gives the product of x and y,  
xy is one variable

To get the constant e use exp(1).

Using the % operator can give confusing results. It always returns the last output from the Kernel, which may have nothing to do with where the cursor is (or which worksheet is active). If Maple doesn't recognize something, it assumes it as a variable; e.g. typing i^2 will give you i2, while we may be wanted -1.

Spaces are optional.

Greek letters may be entered by spelling their name. For example, alpha is always displayed as  $\alpha$ , and Gamma is displayed as  $\Gamma$  (note upper-case).

Built-in Data Capabilities

Maple can handle arbitrary-precision floating point numbers. In other words, Maple can store as many digits for a number as you like, up to the physical limits of your computer's memory. To control this, use the Digits variable.

```
sqrt(2.0);
1.414213562
Digits := 20;
sqrt(2.0);
1.4142135623730950488
```

Maple sets Digits to be 10 by default. You can also temporarily get precision results by calling evalf with a second argument.

```
evalf(sqrt(2), 15);
1.41421356237310
```

Large integers are handled automatically

#### Using symbolic computation

The main feature of Maple is symbolic computation.

In other words, Maple does algebra.

| Example           | Output          | Comments                                                                                                                                    |
|-------------------|-----------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| $(x + y)^2;$      | $(x + y)^2$     | A basic expression.                                                                                                                         |
| $k := x*y + y^2;$ | $k := xy + y^2$ | k is now an alias for the expression. Note that k is simply another name for the expression - they are not equal in the mathematical sense. |
| $p := k/(x+y);$   |                 | You can now use k to refer to the expression. Maple immediately substitutes the value of k.                                                 |
| $k := 'k';$       | k               | You can unassign a variable by assigning it to its own name in single quotes.                                                               |

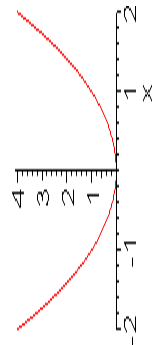
`simplify(p);`      `y`      The `simplify` command does algebraic simplification.

`p := x^2 - 8*x + 15;`      `p := x^2 - 8x + 15`      Maple doesn't mind if you re-use names. The old value is lost.

`solve(p=3,x);`      2,6      Use the `solve` command to solve equations. Note the use of the `=` sign. Here, it is used in a mathematical sense. Maple will try different values for  $x$  until it finds all of them that make the mathematical statement  $x^2 - 8x + 15 = 3$  true.

`dpx := diff(p,x);`      `dpx := 2x - 8`      The `diff` command differentiates an expression with respect to a variable.

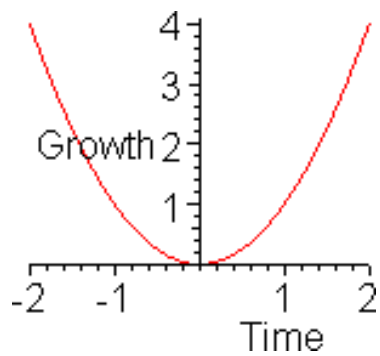
`int(p,x);`



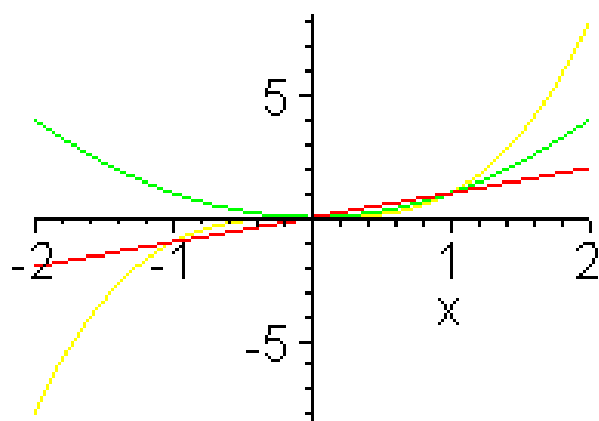
The `int` command integrates an expression. Note that the constant of integration is left off.

### Basic Plotting

Maple can produce graphs very easily. Here are some examples, showcasing the basic capabilities.



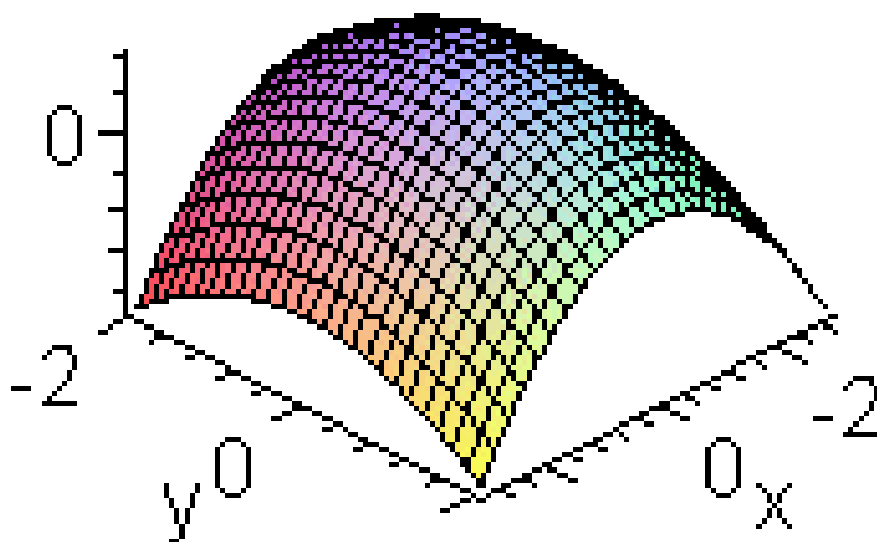
`plot( x^2, x=-2..2);`



`plot( x^2, x=-2..2, y=-10..10);`

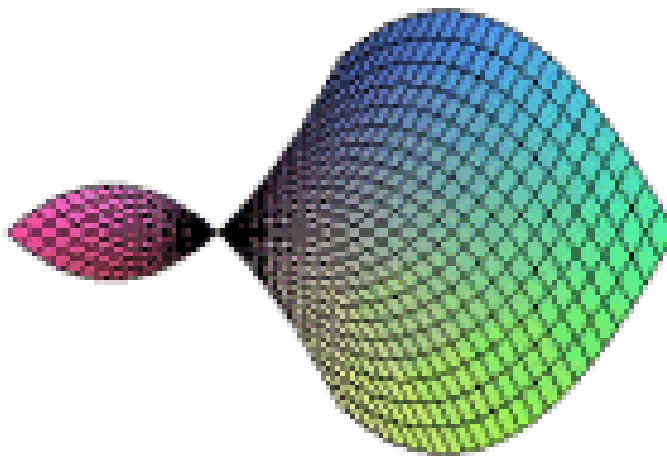
A basic plot.

A plot with vertical axis control.



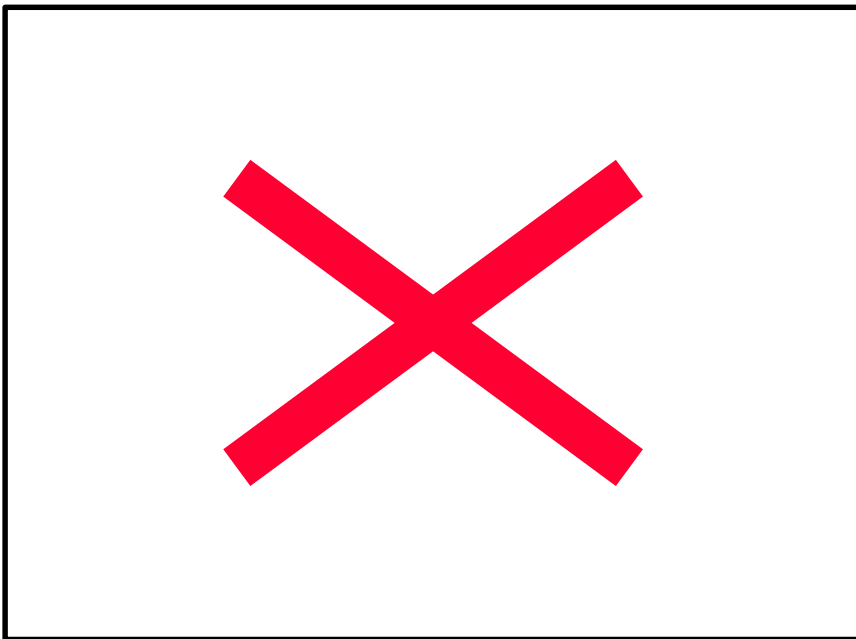
`plot([x, x^2, x^3], x=-2..2);`  
Plot multiple expressions by enclosing them in brackets.

Live



```
plot3d(4-x^2-y^2, x=-3..3, y=-2..2);
```

A basic 3-d plot.



```
smartplot3d(x^2-y^2);
```

Using smartplot to let maple set it's own scaling.

Eigenvals and vectors of a numeric matrix :

Calling Sequence

Eigenvals( A, vecs)

Eigenvals( A, B, vecs)

Parameters

A,B - square matrices of real or complex numbers

vecs - (optional) name to be assigned the matrix of eigenvectors

Example

```
> A := array([[1,2,4],[3,7,2],[5,6,9]]);
```

$$A := \begin{bmatrix} 1 & 2 & 4 \\ 3 & 7 & 2 \\ 5 & 6 & 9 \end{bmatrix}$$

evalf(Eigenvals(A));

$[-.8946025434, 13.74788901, 4.146713483]$

```
> lambda := evalf(Eigenvals(A,vecs));
```

$\lambda := [-.8946025434, 13.74788901, 4.146713483]$

```
> print(vecs);
```

$$\begin{bmatrix} 0.946575123 & 0.3592673648 & -.2987282688 \\ -.2818195168 & 0.4346670837 & 0.7371720079 \\ -.3074361518 & 0.9276415841 & -.6035890728 \end{bmatrix}$$

linalg[eigenvectors] - find the eigenvectors of a matrix

Calling Sequence

eigenvectors( A)

eigenvectors( A, 'radical')

eigenvectors( A, 'implicit')

## Parameters

A - square matrix

The command with(linalg,eigenvalues) allows the use of the abbreviated form of this command.

> with(linalg):

Warning, the protected names norm and trace have been redefined and unprotected

> A := matrix(3,3, [1,-3,3,3, -5,3,6,-6,4]);

$$A := \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

> e := eigenvalues(A);

$e := 4, -2, -2$

> v := [eigenvalues(A)];

$v := [[4, 1, \{[1, 1, 2]\}], [-2, 2, \{[1, 1, 0], [-1, 0, 1]\}]]$

> v[1][1]; # The first eigenvalue

4

> v[1][2]; # Its multiplicity

1

> v[1][3]; # Its eigenvectors

$\{[1, 1, 2]\}$

> v[2][1]; # The second eigenvalue

-2

> v[2][2]; # Its multiplicity

> v[2][2]; # Its multiplicity

2

Help

its worksheet interface

- Waiting for command;
- Restart; refresh memory;
- # #; comments so no action implied

Eval - Evaluate an expression

Calling Sequence

Eval(a, x=n)

Eval(a, {x1=n1,x2=n2,...})

Parameters

a - an expression

x, x1, x2,... - names

n, n1, n2,... - evaluation points

Description

The Eval function is a place holder for evaluation at a point.

The expression a is evaluated at

x = n (x1=n1, x2=n2, ... for the multivariate case).

The call `Eval(a, x=n) mod p` evaluates the polynomial  $a$  at  $x=n$  modulo  $p$ .

Note: The polynomial must be a multivariate polynomial over a finite field.

The call `modp1(Eval(a,n),p)` evaluates the polynomial  $a$  at  $x = n$  modulo  $p$  where  $a$  must be a univariate polynomial in the `modp1` representation, with  $n$  an integer and  $p$  an integer  $> 1$ .

Examples

➤ `Eval(x^2+1,x=3) mod 5;`

0

➤ `Eval(x^2+y,{x=3,y=2}) mod 5;`

> `Eval(int(f(x),x), x=y);`

$\int f(x) dx \Big|_{x=y}$

Eigen values ?;

Solution of Problems

We can use Maple For:

Solution of non-linear equations

by Newton's Method

by Bisection Method

Solution of System of linear equations.

Numerical Integration.

Numerical Solution of ODE's.

Maple performs both numerical and symbolic integration.

Please note that the Maple uses the int function for the both numerical and symbolic integration, but for numerical integration we have to use the additional evalf command

Some inbuilt functions in Maple being used for integration

Numerical Integration

Calling Sequences

`evalf(Int(f, x=a..b))`

`evalf(Int(f, a..b))`

`evalf(Int(f, x=a..b, opts))`

`evalf(Int(f, a..b, opts))`

`evalf(int(f, x=a..b))`

We Define Parameters

$f$  - algebraic expression or procedure; integrand

$x$  - name; variable of integration

$a, b$  - endpoints of the interval of integration

$opts$  - (optional) name or equation of the form

$option=name$ ; options

Description

In the case of a definite integral, which is returned unevaluated, numerical integration can be invoked by applying `evalf` to the unevaluated integral. To invoke numerical integration without

first invoking symbolic integration, use the `inert` function `Int` as in: `evalf( Int(f, x=a..b) )`.

If the integrand  $f$  is specified as a procedure or a Maple operator, then the second argument must be the range  $a..b$  and not an equation. (i.e., a variable of integration must not be specified.)

```
>evalf(Int( exp(-x^3), x = 0..1 ));8075111821
```

```
>evalf(Int( exp(-x^3), x = 0..1 ));8075111821
```

```
>evalf(Int( exp(-x^3), x = 0..1 ));8075111821
```

```
>alg041(); This is Simpson's Method.
```

```
`Input the function F(x) in terms of x`
```

```
`For example:
```

```
> sin (x)
```

```
`Input lower limit of integration and upper limit of integration`
```

```
`separated by a blank`
```

```
> 0 3.14159265359
```

```
`Input an even positive integer N.`
```

```
> 20
```

```
The integral of F from 0.00000000
```

```
to
```

```
3.14159265
```

```
is 2.00000678
```

```
alg041(); This is Simpson's Method.
```

```
`Input the function F(x) in terms of x`
```

```
> x^2
```

```
`Input lower limit of integration and upper limit of integration separated by a blank`
```

```
>0 2
```

```
Input an even positive integer N
```

```
>20
```

```
The integral of F from
```

```
0.00000000
```

```
to
```

```
2.00000000
```

```
is 2.66666667
```

```
> alg041();
```

```
This is Simpson's Method.
```

```
Input the function F(x) in terms of x, for example: cos(x)
```

```
> exp(x-x^2/2)
```

```
Input lower limit of integration and upper limit of integration separated by a blank
```

```
> 0 3.14159265359
```

```
Input an even positive integer N.
```

```
> 20
```

```
The integral of F from
```

```
0.00000000
```

```
to
```

```
3.14159265
```

is 3.41046542

> alg044();

This is Simpson's Method for double integrals.

Input the functions  $F(X,Y)$ ,  $C(X)$ , and  $D(X)$  in terms of  $x$  and  $y$  separated by a space.

For example:  $\cos(x+y) x^3 x$

>  $\exp(y/x) x^3 x^2$

Input lower limit of integration and upper limit of integration separated by a blank

> 0.1 0.5

Input two even positive integers  $N, M$  ; there will be  $N$  subintervals for outer integral and  $M$  subintervals for inner integral - separate with blank

> 10 10

The double integral of  $F$  from

0.100 to 0.500

Is .03330546

obtained with

$N := 10$  and  $M := 10$

> alg045();

`This is Gaussian Quadrature for double integrals.`

`Input the function  $F(x,y)$  in terms of  $x$  and  $y$ `

`For example:  $\sqrt{x^2+y^2}$ `

>  $\exp(y/x)$

Input the functions  $C(x)$ , and  $D(x)$  in terms of  $x$  separated by a space

For example:  $\cos(x) \sin(x)$

>  $x^3 x^2$

Input lower limit of integration and upper limit of integration separated by a blank space

> 0.1 0.5

Input two integers  $M > 1$  and  $N > 1$ . This implementation of Gaussian quadrature requires both to be less than or equal to 5.

$M$  is used for the outer integral and  $N$  for the inner integral - separated by a space.

> 5 5

The double integral of  $F$  from

0.1000 to 0.5000

is 3.3305566120e-02

Or 0.03305566120

obtained with

$M = 5$  and  $N = 5$

## Lecture 44

### Solution of Non-Linear Equations

Bisection Method  
Regula-Falsi Method  
Method of iteration  
Newton - Raphson Method  
Muller's Method  
Graeffe's Root Squaring Method

#### Newton -Raphson Method

An approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Better and successive approximations  $x_2, x_3, \dots, x_n$  to the root are obtained from

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

#### N-R Formula

#### Newton's algorithm

To find a solution to  $f(x)=0$  given an initial approximation  $p_0$

INPUT *initial approximation  $p_0$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$*

OUTPUT *approximate solution  $p$  or message of failure*

Step 1

Set  $i = 1$

Step 2

While  $i \leq N_0$  do Steps 3-6

Step 3

Set  $p = p_0 - f(p_0) / f'(p_0)$

(compute  $p_i$ ).

Step 4

If  $\text{Abs}(p - p_0) < TOL$  OUTPUT ( $p$ );

(The procedure was successful.)

STOP

Step 5 Set  $i = i + 1$

Step 6 Set  $p_0 = p$  (Update  $p_0$ )

Step 7 OUTPUT

(The method failed after  $N_0$  iterations,  $N_0 = 'N_0'$ )

The procedure was unsuccessful

STOP

#### Example

Using Maple to solve a non-linear equation.

$$\cos(x) - x = 0$$

## Solution

The Maple command will be as follows,

Fsolve ( cos (x) -x);

➤ alg023();

➤ This is Newton's Method

Input the function F(x) in terms of x

For example:

> cos(x)-x

Input initial approximation

> 0.7853981635

Input tolerance

> 0.00005

Input maximum number of iterations - no decimal point

> 25

Select output destination

1. Screen

2. Text file

Enter 1 or 2

> 1

Select amount of output

1. Answer only

2. All intermediate approximations

Enter 1 or 2

> 2

## Newton's Method

| I | P           | F(P)           |
|---|-------------|----------------|
| 1 | 0.739536134 | -7.5487470e-04 |
| 2 | 0.739085178 | -7.5100000e-08 |
| 3 | 0.739085133 | 0.0000000e-01  |

Approximate solution = 0.73908513

with F(P) = 0.0000000000

Number of iterations = 3

Tolerance = 5.0000000000e-05

## Another Example

> alg023();

Input the function F(x) in terms of x ,

> sin(x)-1

Input initial approximation

> 0.17853

Input tolerance

> 0.00005

Input maximum number of iterations – no decimal point

> 25

Select output destination

1. Screen

2. Text file

Enter 1 or 2

>2

Select amount of output

1. Answer only

2. All intermediate approximations

Enter 1 or 2

> 2

## Newton's Method

| I | P              | F(P)           |
|---|----------------|----------------|
| 1 | 1.01422964e+00 | -1.5092616e-01 |
| 2 | 1.29992628e+00 | -3.6461537e-02 |
| 3 | 1.43619550e+00 | -9.0450225e-03 |
| 4 | 1.50359771e+00 | -2.2569777e-03 |
| 5 | 1.53720967e+00 | -5.6397880e-04 |
| 6 | 1.55400458e+00 | -1.4097820e-04 |
| 7 | 1.56240065e+00 | -3.5243500e-05 |

More...

|    |                |                |
|----|----------------|----------------|
| 8  | 1.56659852e+00 | -8.8108000e-06 |
| 9  | 1.56869743e+00 | -2.2027000e-06 |
| 10 | 1.56974688e+00 | -5.5070000e-07 |
| 11 | 1.57027163e+00 | -1.3770000e-07 |
| 12 | 1.57053407e+00 | -3.4400000e-08 |
| 13 | 1.57066524e+00 | -8.6000000e-09 |
| 14 | 1.57073085e+00 | -2.1000000e-09 |
| 15 | 1.57076292e+00 | -6.0000000e-10 |

Approximate solution

= 1.57076292

with

$F(P) = 6.0000000000e-10$

Number of iterations = 15

Tolerance =  $5.0000000000e-05$

### Bisection Method

> alg021();

This is the Bisection Method.

Input the function  $F(x)$  in terms of  $x$

For example:

>  $x^3 + 4x^2 - 10$

Input endpoints  $A < B$  separated by blank

> 1 2

Input tolerance

> 0.0005

Input maximum number of iterations - no decimal point

> 25

Select output destination

1. Screen ,

2. Text file

Enter 1 or 2

> 1

Select amount of output

1. Answer only

2. All intermediate approximations

Enter 1 or 2

> 2

### Bisection Method

| I | P              | F(P)           |
|---|----------------|----------------|
| 1 | 1.50000000e+00 | 2.3750000e+00  |
| 2 | 1.25000000e+00 | -1.7968750e+00 |
| 3 | 1.37500000e+00 | 1.6210938e-01  |
| 4 | 1.31250000e+00 | -8.4838867e-01 |
| 5 | 1.34375000e+00 | -3.5098267e-01 |
| 6 | 1.35937500e+00 | -9.6408842e-02 |

|    |                |                |
|----|----------------|----------------|
| 7  | 1.36718750e+00 | 3.2355780e-02  |
| 8  | 1.36328125e+00 | -3.2149969e-02 |
| 9  | 1.36523438e+00 | 7.2030000e-05  |
| 10 | 1.36425781e+00 | -1.6046697e-02 |
| 11 | 1.36474609e+00 | -7.9892590e-03 |

Approximate solution  $P = 1.36474609$   
 with  $F(P) = -.00798926$   
 Number of iterations = 11  
 Tolerance = 5.00000000e-04  
 alg021(); Another example of the Bisection Method.  
 Input the function  $F(x)$  in terms of  $x$ ,  
 > cos(x)  
 Input endpoints  $A < B$  separated by blank  
 > 1 2  
 Input tolerance  
 > 0.0005  
 Input maximum number of iterations - no decimal point  
 > 25  
 Select output destination  
 1. Screen , 2. Text file  
 Enter 1 or 2  
 > 1  
 Select amount of output  
 1. Answer only  
 2. All intermediate approximations  
 Enter 1 or 2  
 > 2

### Bisection Method

|    | P              | F(P)           |
|----|----------------|----------------|
| 1  | 1.50000000e+00 | 7.0737202e-02  |
| 2  | 1.75000000e+00 | -1.7824606e-01 |
| 3  | 1.62500000e+00 | -5.4177135e-02 |
| 4  | 1.56250000e+00 | 8.2962316e-03  |
| 5  | 1.59375000e+00 | -2.2951658e-02 |
| 6  | 1.57812500e+00 | -7.3286076e-03 |
| 7  | 1.57031250e+00 | 4.8382678e-04  |
| 8  | 1.57421875e+00 | -3.4224165e-03 |
| 9  | 1.57226563e+00 | -1.4692977e-03 |
| 10 | 1.57128906e+00 | -4.9273519e-04 |
| 11 | 1.57080078e+00 | -4.4542051e-06 |

Approximate solution  $P = 1.57080078$   
 with  $F(P) = -.00000445$   
 Number of iterations = 11  
 Tolerance = 5.00000000e-04  
 ➤ alg025(); This is the Method of False  
 Position  
 Input the function  $F(x)$  in terms of  $x$   
 > cos(x)-x  
 Input endpoints  $P0 < P1$  separated by a blank space  
 ➤ 0.5 0.7853981635  
 Input tolerance  
 >0.0005  
 Input maximum number of  
 iterations - no decimal point  
 > 25

Select output destination

1. Screen
2. Text file

Enter 1 or 2

>1

Select amount of output

1. Answer only
2. All intermediate approximations

Enter 1 or 2

> 2

## METHOD OF FALSE POSITION

| I | P              | F(P)           |
|---|----------------|----------------|
| 2 | 7.36384139e-01 | 4.51771860e-03 |
| 3 | 7.39058139e-01 | 4.51772000e-05 |
| 4 | 7.39084864e-01 | 4.50900000e-07 |

Approximate solution P = .73908486

with F(P) = .00000045

Number of iterations = 4

Tolerance = 5.00000000e-04

## System of Linear Equations

Gaussian Elimination Gauss-Jordan Elimination

Crout's Reduction Jacobi's

Gauss- Seidal Iteration Relaxation

## Matrix Inversion

> alg061();

This is Gaussian Elimination to solve a linear system.

The array will be input from a text file in the order:

A(1,1), A(1,2), ..., A(1,N+1), A(2,1), A(2,2), ..., A(2,N+1),..., A(N,1), A(N,2), ..., A(N,N+1)

Place as many entries as desired on each line, but separate entries with at least one blank.

Has the input file been created? - enter Y or N.

> y

Input the file name in the form - drive:\name.ext

for example: A:\DATA.DTA

> d:\maple00\dta\alg061.dta

Input the number of equations - an integer.

> 4

Choice of output method:

1. Output to screen
2. Output to text file

Please enter 1 or 2.

> 1

## GAUSSIAN ELIMINATION

The reduced system - output by rows:

```
1.00000000 -1.00000000 2.00000000 -1.00000000 -8.00000000
0.00000000 2.00000000 -1.00000000 1.00000000 6.00000000
0.00000000 0.00000000 -1.00000000 -1.00000000 -4.00000000
0.00000000 0.00000000 0.00000000 2.00000000 4.00000000
```

Has solution vector:

```
-7.00000000 3.00000000 2.00000000 2.00000000
```

with 1 row interchange (s)

> alg071();

This is the Jacobi Method for Linear Systems.

The array will be input from a text file in the order

A(1,1), A(1,2), ..., A(1,n+1), A(2,1), A(2,2), ...,

A(2,n+1), ..., A(n,1), A(n,2), ..., A(n,n+1)

Place as many entries as desired on each line, but separate entries with at least one blank.

The initial approximation should follow in the same format has the input file been created? - enter Y or N.

> y

Input the file name in the form - drive:\name.ext

for example: A:\DATA.DTA

> d:\maple00\alg071.dta

Input the number of equations - an integer.

> 4

Input the tolerance.

> 0.001

Input maximum number of iterations.

> 15

Choice of output method:

1. Output to screen

2. Output to text file

Please enter 1 or 2.

> 1

## JACOBI ITERATIVE METHOD FOR LINEAR SYSTEMS

The solution vector is :

1.00011860 1.99976795

-.99982814 0.99978598

using 10 iterations

with Tolerance 1.0000000000e-03

An approximation to the root is given by

Better and successive approximations  $x_2, x_3, \dots, x_n$  to the root are obtained from

### Example

Using Maple to solve a non-linear equation.

## System of Linear Equations

Input the tolerance.

> 0.001

Input maximum number of iterations.

> 15

Choice of output method:

1. Output to screen

2. Output to text file

Please enter 1 or 2.

> 1

## Summing up

### Non-Linear Equations

Bisection Method (Bolzano)  
Regula-Falsi Method  
Method of iteration  
Newton - Raphson Method  
Muller's Method  
Graeffe's Root Squaring Method

In the method of False Position, the first approximation to the root of  $f(x) = 0$  is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \quad \dots\dots\dots(2.2)$$

Here  $f(x_{n-1})$  and  $f(x_{n+1})$  are of opposite sign. Successive approximations to the root of  $f(x) = 0$  is given by Eq. (2.2).

METHOD OF ITERATION can be applied to find a real root of the equation  $f(x) = 0$  by rewriting the same in the form,  $x = \phi(x)$

$$\left. \begin{array}{l} x_1 = \phi(x_0) \\ x_2 = \phi(x_1) \\ \vdots \\ x_{n+1} = \phi(x_n) \end{array} \right\}$$

In Newton – Raphson Method successive approximations  $x_2, x_3, \dots, x_n$  to the root are obtained from

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

N-R Formula

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, n = 2, 3, \dots$$

**This sequence converges to the root 'b' of  $f(x) = 0$  i.e.  $f(b) = 0$ .**

The Secant method converges faster than linear and slower than Newton's quadratic.

In Muller's Method we can get a better approximation to the root, by using

$$x_{i+1} = x_i + h_i \lambda$$

$$\lambda = \frac{-2f_i \delta_i}{g_i^2 \pm [g_i^2 - 4f_i \delta_i \lambda_i (f_{i-2} \lambda_i - f_{i-1} \delta_i + f_i)]^{1/2}}$$

Where we defined

$$\lambda = \frac{h}{h_i} = \frac{x - x_i}{x_i - x_{i-1}}$$

$$\lambda_i = \frac{h_i}{h_{i-1}}$$

$$\delta_i = 1 + \lambda_i$$

## Systems of Linear Equations

### Gaussian Elimination Gauss-Jordan Elimination Crout's Reduction Jacobi's Gauss-Seidal Iteration Relaxation Matrix Inversion

In Gaussian Elimination method, the solution to the system of equations is obtained in two stages.

- the given system of equations is reduced to an equivalent upper triangular form using elementary transformations
- the upper triangular system is solved using back substitution procedure

Gauss-Jordan method is a variation of Gaussian method. In this method, the elements above and below the diagonal are simultaneously made zero

In Crout's Reduction Method the coefficient matrix [A] of the system of equations is decomposed into the product of two matrices [L] and [U], where [L] is a lower-

triangular matrix and [U] is an upper-triangular matrix with 1's on its main diagonal.

For the purpose of illustration, consider a general matrix in the form

$$[L][U] = [A]$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{12} & l_{22} & 0 \\ l_{13} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Jacobi's Method is an iterative method, where initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way.

In Jacobi's method, the (r + 1)th approximation to the above system is given by Equations

$$\left. \begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(r)} - \dots - \frac{a_{1n}}{a_{11}} x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(r)} - \dots - \frac{a_{2n}}{a_{22}} x_n^{(r)} \\ &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(r)} - \dots - \frac{a_{nn-1}}{a_{nn}} x_{n-1}^{(r)} \end{aligned} \right\}$$

Here we can observe that no element of  $x_i^{(r+1)}$  replaces  $x_i^{(r)}$  entirely for the next cycle of computation.

In Gauss-Seidel method, the corresponding elements of  $x_i^{(r+1)}$  replaces those of  $x_i^{(r)}$  as soon as they become available. It is also called method of Successive Displacement.

The Relaxation Method is also an iterative method and is due to Southwell.

$$dx_i = \frac{R_i}{a_{ii}}$$

**Eigen Value Problems**

## Power Method Jacobi's Method

In Power Method the result looks like

$$u^{(k)} = [A]v^{(k-1)} = q_k v^{(k)}$$

Here,  $q_k$  is the desired largest eigen value and  $v^{(k)}$  is the corresponding eigenvector.

## Interpolation

Finite Difference Operators  
Newton's Forward Difference Interpolation Formula  
Newton's Backward Difference Interpolation Formula  
Lagrange's Interpolation Formula  
Divided Differences  
Interpolation in Two Dimensions  
Cubic Spline Interpolation

## Finite Difference Operators

### Forward Differences

### Backward Differences

### Central Difference

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1},$$

$$i = n, (n-1), \dots, k$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

Thus

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly

$$\nabla y_x = y_x - y_{x-h} = f(x) - f\left(x - \frac{h}{2}\right)$$

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

Shift operator, E

$$E f(x) = f(x+h)$$

$$E^n f(x) = f(x+nh)$$

$$E^n y_x = y_{x+nh}$$

The inverse operator E-1 is defined as

$$E^{-1} f(x) = f(x-h)$$

Similarly,

$$E^{-n} f(x) = f(x-nh)$$

Average Operator,  $\mu$

$$\mu f(x) = \frac{1}{2} \left[ f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right]$$

$$= \frac{1}{2} [y_{x+(h/2)} + y_{x-(h/2)}]$$

Differential Operator, D

$$\left. \begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned} \right\}$$

Important Results

$$\Delta = E - 1$$

$$\nabla = 1 - E^{-1} = \frac{E - 1}{E}$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$hD = \log E$$

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

The Newton's forward difference formula for interpolation, which gives the value of  $f(x_0 + ph)$  in terms of  $f(x_0)$  and its leading differences.

$$\begin{aligned} f(x_0 + ph) &= f(x_0) + p\Delta f(x_0) \\ &+ \frac{p(p-1)}{2!} \Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!} \Delta^3 f(x_0) \\ &+ \dots + \frac{p(p-1)\dots(p-n+1)}{n!} \Delta^n f(x_0) + \text{Error} \end{aligned}$$

An alternate expression is

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \dots + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots + \frac{p(p-1)\dots(p-n+1)}{n!} \Delta^n y_0 + \text{Error}$$

Newton's Backward difference formula is,

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!} \nabla^2 f(x_n) + \frac{p(p+1)(p+2)}{3!} \nabla^3 f(x_n) + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n f(x_n) + \text{Error}$$

Alternatively, this formula can also be written as

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n y_n + \text{Error}$$

Here

$$p = \frac{x - x_n}{h}$$

The Lagrange's formula for interpolation

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} y_i + \dots + \frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

Newton's divided difference interpolation formula can be written as

$$y = f(x) = y_0 + (x-x_0)y[x_0, x_1] + (x-x_0)(x-x_1)y[x_0, x_1, x_2] + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})y[x_0, x_1, \dots, x_n]$$

Where the first order divided difference is defined as

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

## Numerical Differentiation and Integration

We expressed D in terms of  $\Delta$  :

$$D = \frac{1}{h} \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{6} - \frac{\Delta^4}{24} + \frac{\Delta^5}{120} - \dots \right)$$

Using backward difference operator  $\nabla$ , we have

$$hD = -\log(1 - \nabla).$$

On expansion, we have

$$D = \frac{1}{h} \left( \nabla - \frac{\nabla^2}{2} + \frac{\nabla^3}{6} - \frac{\nabla^4}{24} + \dots \right)$$

Using Central difference Operator  $\delta$

$$D = \frac{1}{h} \left( \delta - \frac{\delta^3}{24} + \frac{\delta^5}{640} - \dots \right)$$

Differentiation Using Interpolation

Richardson's Extrapolation

$$\frac{d}{dx} \prod_{i=0}^{n-1} (x - x_i) = \sum_{i=0}^n \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{x - x_i}$$

Thus,  $y'(x)$  is approximated by  $P_n'(x)$  which is given by

$$P_n'(x) = y[x_0, x_1] + [(x - x_1) + (x - x_0)]y[x_0, x_1, x_2] + \dots$$

$$+ \sum_{i=0}^{n-1} \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{x - x_i} y[x_0, x_1, \dots, x_n]$$

$$F_m \left( \frac{h}{2^m} \right) = \frac{4^m F_{m-1} \left( \frac{h}{2^m} \right) - F_{m-1} \left( \frac{h}{2^{m-1}} \right)}{4^m - 1},$$

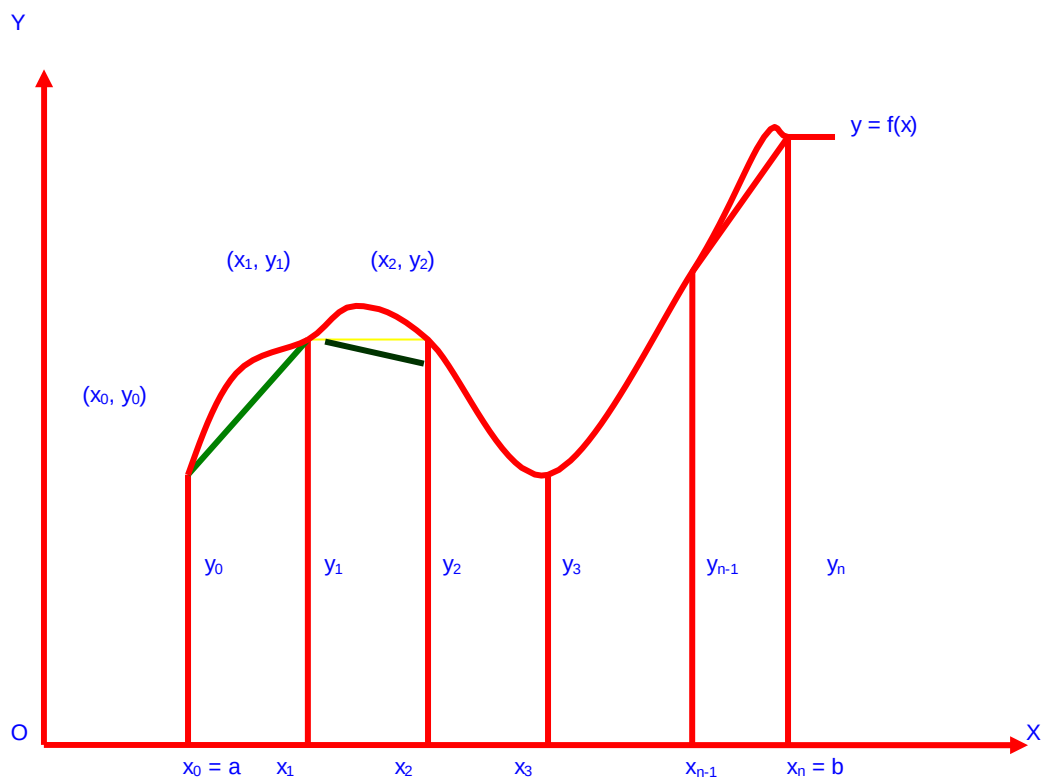
$$m = 1, 2, 3,$$

Basic Issues in Integration

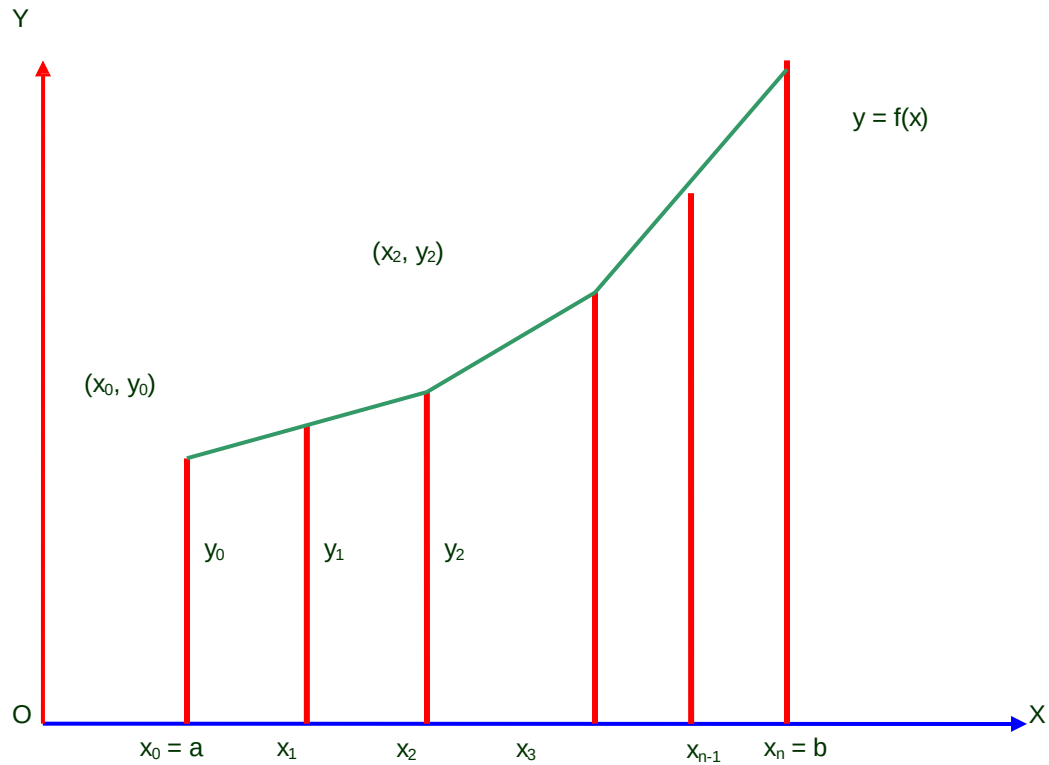
What does an integral represent?

$$\int_a^b f(x) dx = \text{AREA}$$

$$\int_a^b \int_c^d g(x, y) \, dx \, dy = \text{VOLUME}$$



$$\int_{x_0}^{x_3} f(x) dx = \frac{3}{8} h(y_0 + 3y_1 + 3y_2 + y_3) - \frac{3}{80} h^5 y^{(iv)}(\xi)$$



### TRAPEZOIDAL RULE

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n$$

### DOUBLE INTEGRATION

We described procedure to evaluate numerically a double integral of the form

$$I = \int \left[ \int (x, y) dx \right] dy$$

### Differential Equations

Taylor Series

Euler Method

Runge-Kutta Method

Predictor Corrector Method

In Taylor's series we expanded  $y(t)$  by Taylor's series about the point  $t = t_0$  and obtain

$$y(t) = y(t_0) + (t - t_0)y'(t_0) + \frac{(t - t_0)^2}{2!}y''(t_0) + \frac{(t - t_0)^3}{3!}y'''(t_0) + \frac{(t - t_0)^4}{4!}y^{IV}(t_0) + \dots$$

In Euler Method we obtained the solution of the differential equation in the form of a recurrence relation

$$y_{m+1} = y_m + hf(t_m, y_m)$$

We derived the recurrence relation

$$y_{m+1} = y_m + h \left[ \frac{f(t_m, y_m) + f(t_{m+1}, y_{m+1}^{(1)})}{2} \right]$$

Which is the modified Euler's method.

The fourth-order R-K method was described as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_n + h, y_n + k_3)$$

In general, Milne's predictor-corrector pair can be written as

$$\left. \begin{aligned} P: y_{n+1} &= y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) \\ C: y_{n+1} &= y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) \end{aligned} \right\}$$

This is known as Adam's predictor formula.

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) + \frac{251}{720}h^4 \nabla^4 f_n$$

Alternatively, it can be written as

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}] + \frac{251}{720}h^4 \nabla^4 y'_n$$